Name: ______________________________

**Directions:** Show all work. No credit for answers without work. Except when otherwise directed, you may leave answers in terms of binomial/multinomial coefficients, factorials, and sums with a small number of terms.

1. [**4 parts, 6 points each**] Let $U = \{1, 2, 3, 4, 5\}^7$, the set of all lists of length 7 with each entry in $\{1, 2, 3, 4, 5\}$.

   (a) Compute $|U|$.

   (b) Find the number of lists in $U$ that do not have consecutive even integers and do not have consecutive odd integers.

   (c) How many lists in $U$ contain two consecutive entries that are equal?

   (d) If a list in $U$ is chosen at random, what is the probability that the first two entries sum to 8?

2. **[6 points]** How many ways are there to form subsets of size 3 from a set of size $n$? Express your answer as a polynomial in $n$ with simplified coefficients.

3. **[6 points]** How many permutations of $\{1, \ldots, 9\}$ have the even numbers appearing in order? For example $(5, 3, 2, 4, 7, 6, 1, 9, 8)$ counts but $(5, 3, 6, 4, 7, 2, 1, 9, 8)$ does not.

4. A fair coin is flipped 8 times.

   (a) **[4 points]** Give the sample space $\Omega$. Then compute $|\Omega|$.

   (b) **[6 points]** What is the probability that there are equally many heads and tails?

   (c) **[6 points]** What is the probability that there is at least one head in the first four flips and at least one head in the last four flips?

5. **[2 parts, 6 points each]** A standard deck of cards has one card for each of the suit/rank pairs. The suits are spades, hearts, diamonds, and clubs; the ranks are ace, 2 through 10, jack, queen, and king. Four hands with 13 cards each are dealt from a freshly shuffled deck to players $A$, $B$, $C$, and $D$.

   (a) What is the probability that player $A$ gets all the spades?

   (b) What is the probability that players $A$ and $B$ together get all the spades and players $C$ and $D$ get none?

6. **[6 points]** How many non-negative integer solutions are there to $x_1 + x_2 + \cdots + x_6 = 83$ such that $x_1 \leq 25$?

7. **[6 points]** How many ways can we distribute 15 identical red balls and 4 identical green balls into 3 labeled boxes? (Putting all 15 red balls in box 1 and 2 green balls each in boxes 2 and 3 is different prom putting all 15 red balls in box 3 and putting 2 green balls in boxes 1 and 2.)

8. **[10 points]** Give a combinatorial proof that $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$.

9. **[2 parts, 6 points each]** Let $A$ be the set of all lists in $\{1,2,3,4,5\}^n$ where every even entry appears before every odd entry, and let $B$ be the set of all lists in $\{a,b,c\}^{n+1}$ that contain at least one $c$. For example, if $n=4$, then $(4,2,3,3) \in A$ and $(a,c,b,c,c) \in B$.

   (a) Describe a bijection $f\colon A \to B$. For $n=4$, explicitly compute $f(4,2,3,3)$ and find the element in $A$ that $f$ maps to $(a,c,b,c,c)$.

   (b) Use the bijection to find a simple formula for $|A|$.