Name: Solutions

Directions: Show all work. No credit for answers without work. Except when otherwise directed, you may leave answers in terms of binomial/multinomial coefficients, factorials, and sums with a small number of terms.

1. **[4 parts, 6 points each]** Let $U = \{1, 2, 3, 4, 5\}^7$, the set of all lists of length 7 with each entry in $\{1, 2, 3, 4, 5\}$.

   (a) Compute $|U|$.

   $\underline{5}\ \underline{5}\ \cdots\ \underline{5}$

   $|U| = \boxed{5^7}$

   (b) Find the number of lists in U that do not have consecutive even integers and do not have consecutive odd integers.

   These lists must alternate even/odd/even or odd/even/odd...

   Rule of sum:

   $\underline{e}\ \underline{o}\ \underline{e}\ \underline{o}\ \underline{e}\ \underline{o}\ \underline{e}$

   e: $\{2,4\}$ 2^4; o: $\{1,3,5\}$ 3^3 → $2^4 \cdot 3^3$

   $\underline{o}\ \underline{e}\ \underline{o}\ \underline{e}\ \underline{o}\ \underline{e}\ \underline{o}$

   e: $\{2,4\}$ 2^3; o: $\{1,3,5\}$ 3^4

   Total: $\boxed{2^4 \cdot 3^3 + 2^3 \cdot 3^4}$

   $= 6^3[2+3] = \boxed{6^3 \cdot 5}$

   (c) How many lists in U contain two consecutive entries that are equal?

   Count complement; # with no two consecutive elts equal:

   ① Choose first — 5 opts

   ② through ⑦: Choose a value different from the previous value — 4 opts

   $\_\ \_\ \_\ \_\ \_\ \_\ \_$

   So # with no two consecutive equal $= 5 \cdot 4^6$.

   Total with two consecutive equal $= |U| - 5 \cdot 4^6$

   $= 5^7 - 5 \cdot 4^6$

   $= \boxed{5(5^6 - 4^6)}$

   (d) If a list in U is chosen at random, what is the probability that the first two entries sum to 8?

   $\_\ \_\ \_\ \_\ \_\ \_\ \_$

   Let A be the lists in U with first two entries summing to 8.

   To count A with rule of product:

   ① Choose first two entries from $(3,5), (4,4), (5,3)$ [3 opts]

   $\underline{5}\ \underline{3}\ \_\ \_\ \_\ \_\ \_$

   ② Choose next 5 entries, no restrictions — 5^5 opts.

   So $Pr(A) = \frac{|A|}{|U|} = \frac{3 \cdot 5^5}{5^7}$

   $= \frac{3}{5^2} = \boxed{\frac{3}{25}} = \boxed{12\%}$

2. **[6 points]** How many ways are there to form subsets of size 3 from a set of size n? Express your answer as a polynomial in n with simplified coefficients.

$$\binom{n}{3} = \frac{n_{(3)}}{3!} = \frac{n(n-1)(n-2)}{6} = \frac{1}{6}n\left[(n-1)(n-2)\right] = \frac{1}{6}n\left[n^2 - 3n + 2\right]$$

$$= \boxed{\frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n}$$

3. **[6 points]** How many permutations of $\{1, \ldots, 9\}$ have the even numbers appearing in order? For example $(5, 3, 2, 4, 7, 6, 1, 9, 8)$ counts but $(5, 3, 6, 4, 7, 2, 1, 9, 8)$ does not.

① Choose 4 of the 9 positions for the even numbers and fill them in order $\binom{9}{4}$ opts

_ 2 4 _ _ 6 _ 8 _

② Fill in remaining 5 positions with a permutation of $\{1,3,5,7,9\}$ ($5!$ opts).

So total is $\boxed{\binom{9}{4}5!} = \frac{9!}{4!\,5!}\cdot 5! = \boxed{\frac{9!}{4!}} = \boxed{9_{(5)}}$

4. A fair coin is flipped 8 times.

(a) **[4 points]** Give the sample space Ω. Then compute $|\Omega|$.

$\Omega = \{H,T\}^8$, $|\Omega| = 2^8$ (2 options for each of 8 flips).

(b) **[6 points]** What is the probability that there are equally many heads and tails?

- $A = \{L \in \Omega : L \text{ has 4 H entries}\}$.
- Now $|A| = \binom{8}{4}$ (choose 4 positions for H, fill the rest with T).

So $Pr(A) = \frac{|A|}{|\Omega|} = \boxed{\frac{\binom{8}{4}}{2^8}}$

(c) **[6 points]** What is the probability that there is at least one head in the first four flips and at least one head in the last four flips?

- $B = \{L \in \Omega : L$ has an H in the first 4 spots and an H in the last 4 spots$\}$.
- To count B:

① Fill in first 4 slots with a sequence of H and T containing ≥ 1 H: ($2^4 - 1$ options)

② Fill in last 4 slots similarly ($2^4 - 1$ options)

- $|B| = (2^4-1)(2^4-1)$

So $Pr(B) = \frac{|B|}{|\Omega|} = \boxed{\frac{(2^4-1)^2}{2^8}} = \frac{2^8 - 2^5 + 1}{2^8}$

$= \frac{256-31}{256} = \boxed{\frac{225}{256}}$

5. **[2 parts, 6 points each]** A standard deck of cards has one card for each of the suit/rank pairs. The suits are spades, hearts, diamonds, and clubs; the ranks are ace, 2 through 10, jack, queen, and king. Four hands with 13 cards each are dealt from a freshly shuffled deck to players A, B, C, and D.

   (a) What is the probability that player A gets all the spades?

• Ω = ways to distribute 13 cards each to A, B, C, D; $|\Omega| = \binom{52}{13,13,13,13}$.

• Let E be subset of Ω in which A gets all 13 spades.

• To count E, distribute all 39 non-spades to B, C, D; there are $\binom{39}{13,13,13}$ ways

• $Pr(E) = \dfrac{\binom{39}{13,13,13}}{\binom{52}{13,13,13,13}} = \dfrac{39!}{(13!)^3} \cdot \dfrac{(13!)^4}{52!} = \dfrac{39! \cdot 13!}{52!} = \dfrac{1}{\binom{52}{13}}$

   (b) What is the probability that players A and B together get all the spades and players C and D get none?

• Let F be the event that A and B get all spades. To count F: ① distribute 39 non-spades to team A/B, player C, and player D: $\binom{39}{13,13,13}$ options
② of the 26 cards for team A/B, choose 13 for A $\binom{26}{13}$ opts.

$Pr(F) = \dfrac{\binom{39}{13,13,13} \cdot \binom{26}{13}}{\binom{52}{13,13,13,13}} = \dfrac{39!}{(13!)^3} \cdot \dfrac{26!}{(13!)^2} \cdot \dfrac{(13!)^4}{52!} = \dfrac{39!\, 26!}{13!\, 52!} = \dfrac{\binom{26}{13}}{\binom{52}{13}} = \dfrac{\binom{39}{13}}{\binom{52}{26}}$

choose 13 positions to be spades in the deck

choose 26 cards for players A and B.

6. **[6 points]** How many non-negative integer solutions are there to $x_1 + x_2 + \cdots + x_6 = 83$ such that $x_1 \leq 25$?

Count complement.

⇒ All solns: 83 stars, 5 bars ⇒ $\binom{88}{5}$

⇒ solns with $x_1 \geq 26$: $(x_1 - 26) + x_2 + \cdots + x_6 = 57$

$y_1 + y_2 + \cdots + y_6 = 57$

So 57 stars, 5 bars ⇒ $\binom{62}{5}$

Total solns: $\binom{88}{5} - \binom{62}{5}$

7. **[6 points]** How many ways can we distribute 15 identical red balls and 4 identical green balls into 3 labeled boxes? (Putting all 15 red balls in box 1 and 2 green balls each in boxes 2 and 3 is different prom putting all 15 red balls in box 3 and putting 2 green balls in boxes 1 and 2.)

① Distribute red balls; #ways equals # solns to

$x_1 + x_2 + x_3 = 15$

⇒ 15 stars, 2 bars ⇒ $\binom{17}{2}$ options

② Distribute green balls

$y_1 + y_2 + y_3 = 4$ ⇒ 4 stars, 2 bars ⇒ $\binom{6}{2}$ options

Total # ways $= \binom{17}{2} \cdot \binom{6}{2}$

8. **[10 points]** Give a combinatorial proof that $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$.

Let $A = \binom{[n]}{k}$, let $A_1 = \{X \in A : n-1, n \notin X\}$, $A_2 = \{X \in A : n-1 \notin X, n \in X\}$, $A_3 = \{X \in A : n-1 \in X, n \notin X\}$ and $A_4 = \{X \in A : n-1, n \in X\}$. Now $\{A_1, \ldots, A_4\}$ is a partition of A and so $|A| = |A_1| + |A_2| + |A_3| + |A_4|$. Also, $|A| = \binom{n}{k}$.

Now $|A_1| = \binom{n-2}{k}$ (since all k elements are chosen from $[n-2]$), $|A_2| = |A_3| = \binom{n-2}{k-1}$ (since $k-1$ elts are chosen from $[n-2]$) and $|A_4| = \binom{n-2}{k-2}$, since $k-2$ elts are chosen from $[n-2]$. It follows that $\binom{n}{k} = |A| = \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k-2}$. ∎

9. **[2 parts, 6 points each]** Let A be the set of all lists in $\{1,2,3,4,5\}^n$ where every even entry appears before every odd entry, and let B be the set of all lists in $\{a,b,c\}^{n+1}$ that contain at least one c. For example, if $n = 4$, then $(4,2,3,3) \in A$ and $(a,c,b,c,c) \in B$.

   (a) Describe a bijection $f\colon A \to B$. For $n = 4$, explicitly compute $f(4,2,3,3)$ and find the element in A that f maps to (a,c,b,c,c).

Given $(x_1, \ldots, x_n) \in A$, let l be the number of even entries. The function f on input $(x_1, \ldots, x_n)$ outputs $(y_0, \ldots, y_n)$ where we set $y_l = c$, $y_j = \begin{cases} a & \text{if } x_j = 2 \\ b & \text{if } x_j = 4 \end{cases}$ for $j < l$, and $y_j = \begin{cases} a & \text{if } x_j = 1 \\ b & \text{if } x_j = 3 \\ c & \text{if } x_j = 5 \end{cases}$ for $j > l$. Note that y_l is the first entry in $(y_0, \ldots, y_n)$ with value c, and so the first c signals the transition between even entries and odd entries.

In particular, $f(4,2,3,3) = (b,a,c,b,b)$ and f maps $(2,3,5,5)$ to (a,c,b,c,c).

   (b) Use the bijection to find a simple formula for $|A|$.

Since f is a bijection, we have $|A| = |B|$. We can count B more easily. We count the complement. There are 3^{n+1} sequences $(y_0, \ldots, y_1) \in \{a,b,c\}^{n+1}$; of these, 2^{n+1} do not contain a c. Therefore $|A| = |B| = \boxed{3^{n+1} - 2^{n+1}}$. ∎