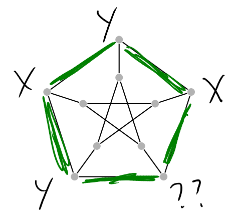
Name: Solutions.

Directions: Show all work.

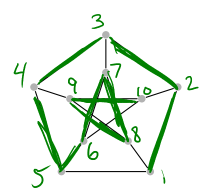
1. **[3 parts, 2 points each]** Let G be the Petersen graph. Recall that $V(G)$ is the set of 2-element subsets of $\{1,2,3,4,5\}$ with $uv \in E(G)$ if and only if u and v are disjoint.



(a) Is G bipartite? Prove your answer is correct.

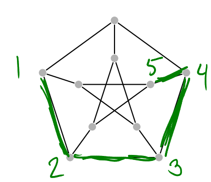
$\boxed{\text{No}}$, G is not bipartite. Note that G contains a 5-cycle, as shown above. If G were bipartite, then the parts must alternate along the vertices of every cycle, and so all cycles must be even. Since G has an odd cycle, G is not bipartite.

(b) Recall that P_n is the path with n vertices. Let k be the maximum integer such that P_k is a subgraph of G. Determine k and find a copy of P_k as a subgraph of G. No proof required.



Since G contains a copy of P_{10} (as shown above), $k \geq 10$. Since $V(G) = 10$, we have $k \leq 10$. Therefore $k = 10$.

(c) Let t be the maximum integer such that P_t is an *induced* subgraph of G. Determine t and find a copy of P_t as an *induced* subgraph of G. No proof required.



Since G contains an induced copy of P_5 (as shown above), $t \geq 5$.

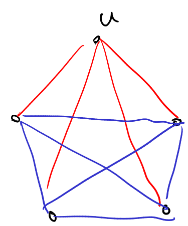
Proof not required

To see that G contains no induced copy of P_6, we can use the algebraic definition of G. Without loss of generality, an induced P_6 must begin as follows: $\underline{12}$ $\underline{34}$ $\underline{15}$ $\underline{24}$ $\underline{13}$. The last vertex u is a 2-elt subset of $\{2,4,5\}$ and must intersect all of $\{1,2\}$, $\{3,4\}$, and $\{1,5\}$, which is impossible since $|u| = 2$. Therefore G has no induced copy of P_6 and $t \leq 5$.

2. **[2 parts, 2 points each]** Graph Ramsey Problems.

   (a) Prove that $K_5 \not\to (P_5, P_5)$.

Pf.

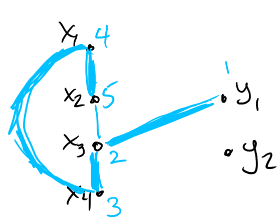


Let H be the 2-edge-coloring of K_5 with a vertex u incident to all red edges and the remaining 4 vertices inducing a blue copy of K_4. The blue subgraph is a copy of K_4 plus an isolated vertex and therefore does not contain P_5. The red subgraph is $K_{1,4}$ and does not even contain a copy of P_4. Since there exists a {blue, red}-edge-coloring of K_5 avoiding monochromatic copies of P_5, we have $K_5 \not\to (P_5, P_5)$. ∎

   (b) Recall that C_n is the n-vertex cycle. Use the fact that $K_6 \to (C_4, C_4)$ to prove that $K_6 \to (P_5, P_5)$.

Let H be a {blue, red}-edge-coloring of K_6 and suppose for a contradiction that H has no monochromatic copies of P_5. Since $K_6 \to (C_4, C_4)$, we have that H contains a monochromatic 4-cycle $x_1 x_2 x_3 x_4$; we may assume without loss of generality that $x_1 x_2 x_3 x_4$ is blue. Let y_1, y_2 be the other two vertices in H. If y_i has a blue neighbor x_j then $y_i x_j$ plus traveling around the other vertices in the 4-cycle $x_1 x_2 x_3 x_4$ gives a blue copy of P_5, as shown in Figure 1. Therefore all edges of the form $y_i x_j$ are red, as shown in Figure 2.

Fig 1



But now $x_1 y_1 x_2 y_2 x_3$ is a red copy of P_5, a contradiction. Since every 2-edge-coloring of K_6 contains a monochromatic copy of P_5, we have $K_6 \to (P_5, P_5)$. ∎

Fig 2