Name: Solutions

Directions: Show all work.

1. Let $U = [n]^2 = \{(x, y) \colon x, y \in \{1, \dots, n\}\}$.

   (a) **[4 points]** Use U to give a combinatorial proof that $n^2 = 2\binom{n}{2} + n$.

Note that $|U| = n^2$ since there are n choices for both x and y. Let $A_< = \{(x,y) \in U : x < y\}$, let $A_> = \{(x,y) \in U : x > y\}$ and $A_= = \{(x,y) \in U : x = y\}$. Clearly, $\{A_<, A_>, A_=\}$ is a partition of U. Note that $|A_<| = |A_>| = \binom{n}{2}$, since once a pair of distinct elements from $[n]$ are chosen, there is one way to order them to get an element in $A_<$ and the other way gives an element in $A_>$. Also, $|A_=| = n$ since a single value from $[n]$ is chosen and repeated to give an element in $A_=$. So

$$n^2 = |U| = |A_<| + |A_>| + |A_=| = \binom{n}{2} + \binom{n}{2} + n = 2\binom{n}{2} + n.$$ ∎

   (b) **[3 points]** Use part (a) and an identity from HW12 to give a formula for $\sum_{k=1}^{n} k^2$.

We compute $\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n}\left(2\binom{k}{2} + k\right) = 2\sum_{k=1}^{n}\binom{k}{2} + \sum_{k=1}^{n}\binom{k}{1}$. Recall from HW12, we have $\sum_{t=k}^{n}\binom{t}{k} = \binom{n+1}{k+1}$; relabeling, this becomes $\sum_{k=r}^{n}\binom{k}{r} = \binom{n+1}{r+1}$. With $r=2$, we get $\sum_{k=2}^{n}\binom{k}{2} = \binom{n+1}{3}$ and with $r=2$, we get $\sum_{k=1}^{n}\binom{k}{1} = \binom{n+1}{2}$. So $\sum_{k=1}^{n} k^2 = 2\sum_{k=1}^{n}\binom{k}{2} + \sum_{k=1}^{n}\binom{k}{1} = \boxed{2\binom{n+1}{3} + \binom{n+1}{2}}$. ∎

2. **[3 points]** Let n be a positive integer. Let A be the set of all triples $\{x, y, z\} \in \binom{[n]}{3}$ such that $x < y < z$ and $z - y = y - x$. Let B be the set of all pairs $\{a, b\} \in \binom{[n]}{2}$ such that $a < b$ and $b - a$ is even. Give a bijection to show that $|A| = |B|$.

We map $\{x,y,z\} \in A$ with $x < y < z$ to $\{x, z\} \in B$. Note that since $z - y = y - x$, we have $z - x - y = y - 2x$ and so $z - x = 2y - 2x = 2(y - x)$. This means that $\{x, z\}$ is an element in B.

To see that this map is a bijection, consider $\{a, b\} \in B$ with $a < b$. Any elt in A that maps to B must have its smallest element x equal to a and its largest element z equal to b. Also, for $\{x,y,z\} \in A$, we have $z - y = y - x$ or $y = \frac{x+z}{2} = x + \frac{z-x}{2}$. So y is also determined. Hence every element in B is mapped to by one element in A. So our map is a bijection. ∎