Directions: Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. [5.4] Let C_n be the nth Catalan number. Recall that $C_n = \frac{1}{n+1}\binom{2n}{n}$ and C_n satisfies the recurrence $C_0 = 1$ and $C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}$ for $n \geq 1$.

   Let A_n be the set of permutations $(x_1, \ldots, x_n)$ of $[n]$ that do not contain three entries x_i, x_k, x_j such that $i < k < j$ and $x_i < x_j < x_k$. Let $a_n = |A_n|$. Our aim is to prove that the sequences $(a_0, a_1, \ldots)$ and $(C_0, C_1, \ldots)$ are the same sequence, meaning that $a_n = C_n$.

   (a) Let k and n be integers such that $1 \leq k \leq n$. Let $A_{n,k}$ be the set of permutations $(x_1, \ldots, x_n) \in A_n$ such that $x_k = n$. Prove that $|A_{n,k}| = a_{k-1}a_{n-k}$.

   (b) Let n be a positive integer. Prove that $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$.

   (c) Explain why this implies that $a_n = C_n$ for all n.

2. [8.1] Use inclusion/exclusion to find a formula for the number of surjective functions f from $\{1, \ldots, n\}$ onto $\{1, 2, 3, 4\}$.

3. [10.1.13] Prove that $d_1, \ldots, d_p$ is graphic if and only if $p - 1 - d_1, \ldots, p - 1 - d_p$ is graphic.

4. [10.2.1] Let x and y be vertices of a graph.

   (a) Suppose that there is a closed walk containing both x and y. Must there be a closed trail containing both x and y?

   (b) Suppose that there is a closed trail containing both x and y. Must there be a cycle containing both x and y?