Math571 **Homework 6** **Due May 2, 2025**

Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [12.1.19] *Another proof of Dilworth's Theorem.* The poset on the right below arises from that on the left by deleting the central covering pair. All other comparabilities remain.

   (a) Suppose that x covers y and z in P. Let Q and R be the posets obtained by deleting (y, x) and (z, x), respectively, from the set of relations in P. Prove that $\min\{w(Q), w(R)\} = w(P)$. (Hint: take maximum antichains in Q and R, and consider the maximal elements and minimal elements in the union of these antichains as a subposet of P.)

   (b) Use part (a) to prove Dilworth's Theorem. (Harzheim [1983])

2. [12.2.11] A *semiantichain* in $P \times Q$ is a subset having $(u, v) < (u', v')$ for two elements of S only if $u < u'$ and $v < v'$. A product poset has the 2-*part Sperner property* when some single rank is a largest semiantichain.

   (a) Prove that the product of two symmetric chain orders is 2-part Sperner.

   (b) Use part (a) to show that for any 2-coloring of the elements of $[n]$, the largest subposet of $\underline{2}^n$ having no related pair with monochromatic difference consists of a largest rank.

3. [12.2.16] A *universal subset list* on an alphabet S is a word having every subset of S as a consecutive substring. For example, 1231 is such a list on [3]. Listing all 2^n subsets consecutively yields a universal subset list on $[n]$ with length $n2^{n-1}$, since the average size is $n/2$.

   (a) For even n, use symmetric chain decompositions of two copies of $\underline{2}^{n/2}$ to construct a universal subset list on $[n]$ with length asymptotically at most $(4/\pi)2^n$. (Hint: Use Stirling's formula to approximate $\binom{k}{\lfloor k/2 \rfloor}$.)

   (b) Prove that a universal subset list on $[n]$ has size at least $c2^n/\sqrt{n}$ for some positive constant c.

   Comment: using that the average size of a chain in a symmetric chain decomposition of $\underline{2}^k$ is $2^k/\binom{k}{\lfloor k/2 \rfloor}$ or $O(\sqrt{k})$, and hence the average size of a top element is $k/2 + O(\sqrt{k})$, improves the bound in part (a) by a factor of two.

4. [12.2.21] Two chain partitions are *orthogonal* if no two elements appear in the same chain in both partitions.

   (a) Define $\mathcal{D}$ from the bracketing decomposition $\mathcal{C}$ of $\underline{2}^n$ by changing each set to its complement and reversing the order of chains. Prove that for $n \geq 4$, a slight change in $\mathcal{D}$ yields a chain partition orthogonal to $\mathcal{C}$.

   (b) Construct two orthogonal Dilworth decompositions of for each of $\underline{2}^2$ and $\underline{2}^3$, and construct three pairwise orthogonal Dilworth decompositions of $\underline{2}^4$.

   (c) Prove that $\underline{2}^n$ has at most $\lceil (n+1)/2 \rceil$ pairwise orthogonal Dilworth decompositions.

5. [12.2.17] *The Littlewood-Offord Problem.* Let $a_1, \dots, a_n$ be vectors in $\mathbb{R}^d$, each having length at least 1. Let $R_1, \dots, R_k$ be regions in $\mathbb{R}^d$, each having diameter less than 1 (i.e. contained in the interior of a sphere of diameter 1), and let R be their union. Let $d_k(n) = \sum_{i=r}^{s} \binom{n}{i}$, where $r = \lfloor (n-k+1)/2 \rfloor$ and $s = \lfloor (n+k-1)/2 \rfloor$ (this counts the k middle ranks in $\underline{2}^n$).

   (a) Prove that $d_k(n) = d_{k+1}(n-1) + d_{k-1}(n-1)$.

   (b) Prove that the number of $\{0,1\}$-vectors x such that $\sum x_i a_i \in R$ is at most $d_k(n)$. (Hint: to apply part (a) in an inductive proof, one must group these vectors x into sets corresponding to two problems of the same type with $n-1$ vectors, one having $k+1$ regions and one having $k-1$ regions.)

6. [12.2.27] Let Π_n denote the poset of partitions of $[n]$, with $\sigma \leq \tau$ if σ is a union of partitions of the blocks of τ. Partitions with k blocks have rank $n-k$, so $N_{n-k}(\Pi_n) = S(n,k)$ (the Sterling number). For even n, let A be the set of partitions of $[n]$ into two blocks of size $n/2$. Use A and its shadow to prove that Π_n with n even is not an LYM order when $n \geq 20$. (Comment: Rota asked whether Π_n always has the Sperner property; Canfield [1978] showed that it doesn't. Shearer [1979] and Jichang–Kleitman [1984] reduced the least n such that Π_n is not Sperner to 4×10^9 and then to 3.4×10^6.