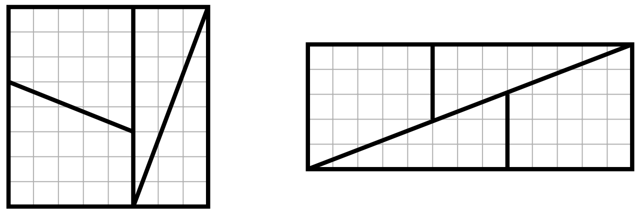
Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines that apply to all homeworks.

1. Prove that $F_n^2 = F_{n-1}F_{n+1} + (-1)^n$ for $n \geq 1$. Manipulate the identity to explain why Lewis Carroll's "proof" below that $64 = 65$ (and larger analogues) seems reasonable.



2. Generating functions. Let $A(x)$ be the generating function for the Fibonacci sequence F_n, with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
   (a) Obtain $A(x)$ from the Fibonacci recurrence.
   (b) Obtain $A(x)$ by building it combinatorially (without the recurrence), using the model that F_n is the number of $\{1, 2\}$-lists that sum to n.
   (c) Expand the generating function to prove that $F_n = \sum_{k \geq 0} \binom{n-k}{k}$.

3. [CM 2.3.8] In the game "High/low", a number is chosen uniformly at random from $[n]$, where n is known. The player makes a guess, winning the prize of correct. Otherwise, she learns whether the guess is high or low and guesses again. When each guess is made randomly from the remaining possible values, the expected numbers of guesses when $n \in \{0, 1, 2, 3\}$ are $0, 1, 3/2, 17/9$, respectively.

   The *harmonic number* H_n is defined by $H_n = \sum_{k=1}^{n} 1/k$. Obtain a formula for the expected number of guesses made to win the prize, using one harmonic number in the formula. (Hint: Obtain a recurrence with many terms. Simplify via natural substitutions and reduce to a first-order recurrence.)

4. Let $s(n)$ be the number of sequences $(x_1, \ldots, x_k)$ of integers satisfying $1 \leq x_i \leq n$ for all i and $x_i \geq 2x_{i-1}$ for $1 < i \leq k$. (The length of the sequence is not specified, and the empty sequence is included, and therefore $s(0) = 1$.)
   (a) Prove that $s(n) = s(n-1) + s(\lfloor n/2 \rfloor)$ for $n \geq 1$.
   (b) Let $S(t) = \sum_{n \geq 0} s(n)t^n$, so that $S(t)$ is the generating function for the sequence $\langle s \rangle$. Show that $(1-t)S(t) = (1+t)S(t^2)$.

   Medium Challenge: determine good bounds on $s(n)$.

5. Let a_n be the number of domino tilings of a $(2 \times n)$-rectangle and let b_n be the number of domino tilings of a $(3 \times 2n)$-rectangle. Obtain bounded order recurrences for $\langle a \rangle$ and $\langle b \rangle$. (Medium challenge: find a bounded order recurrence for the number of domino tilings of a $(4 \times n)$-rectangle.)

6. [CM 2.2.34] The Delannoy numbers satisfy $d_{m,n} = d_{m,n-1} + d_{m-1,n-1} + d_{m-1,n}$ for $m, n \geq 1$ with $d_{m,0} = d_{0,n} = 1$ for $m, n \geq 0$. Find the generating function $\sum_{m,n \geq 0} d_{m,n}x^m y^n$. Using a clever factorization of the denominator, prove that $d_{m,n} = \sum_k 2^k \binom{m}{k}\binom{n}{k}$. (Hint: factor $(1-x)(1-y)$ out of the denominator.)