**Directions:** Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Prove the first three identities below by counting a set in two ways. In each case, give a single direct argument without manipulating the formulas. In part (d), find a closed form solution for the sum and give a combinatorial proof.

   (a) $\binom{2n}{n} = 2\binom{2n-1}{n-1}$

   (b) $\sum_k \binom{k}{l}\binom{n}{k} = \binom{n}{l}2^{n-l}$

   (c) $\sum_{i=1}^{n} i(n-i) = \sum_{i=1}^{n}\binom{i}{2}$

   (d) $\sum_{j=1}^{m}(m-j)2^{j-1}$

2. The graph $P_n \,\square\, P_3$ is the *Cartesian product* of the $n$-vertex path and the 3-vertex path; it has vertex set $\{(x,y)\colon\ x \in [n] \text{ and } y \in [3]\}$ with $(x,y)$ and $(x',y')$ adjacent if and only if (1) $x = x'$ and $|y-y'| = 1$, or (2) $|x-x'| = 1$ and $y = y'$. An *independent set* is a set of pairwise non-adjacent vertices.

   Let $s_0 = 1$ and for $n \ge 1$, let $s_n$ be the number of independent sets in $P_n \,\square\, P_3$. Let $X = \{(n,1),(n,2),(n,3)\}$; these are the vertices in the last column of $P_n \,\square\, P_3$. For $n \ge 1$, let $a_n$, $b_n$, $c_n$, and $d_n$ be the number of independent sets $S$ in $P_n \,\square\, P_3$ such that $S \cap X$ equals $\varnothing$, equals $\{(n,1)\}$ or $\{(n,3)\}$, equals $\{(n,2)\}$, or equals $\{(n,1),(n,3)\}$, respectively. Also, let $x_n$ be the column vector $[a_n\ b_n\ c_n\ d_n]^T$ for $n \ge 1$, and for convenience define $x_0 = [a_0\ b_0\ c_0\ d_0]^T = [1\ 0\ 0\ 0]^T$.

   (a) Find a $4 \times 4$-matrix $A$ such that $x_n = Ax_{n-1}$ for $n \ge 1$.

   (b) Using that $s_n = [1\ 1\ 1\ 1]x_n = [1\ 1\ 1\ 1]A^n x_0$, find row vectors $z_0, \ldots, z_4$ such that $s_{n+j} = z_j x_n$ for $0 \le j \le 4$.

   (c) Let $B$ be the $5 \times 4$-matrix with rows $z_0, \ldots, z_4$. Find a nonzero row vector $y$ such that $yB = 0$.

   (d) Using that $yBx_n = 0$, find a 4th order homogeneous linear recurrence for $s_n$.

3. By pairing positive and negative contributions, give a combinatorial proof for

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\binom{n}{n/2} & \text{if } n \text{ is even} \end{cases}.$$

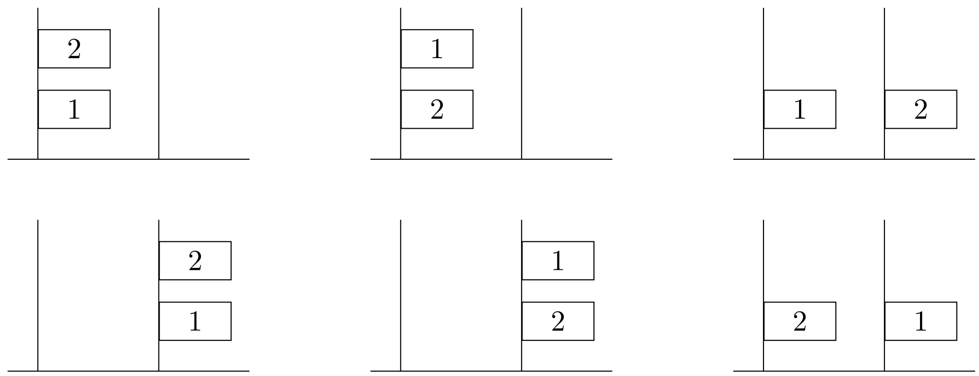
4. Give a combinatorial proof for the following identity by devising a set that both sides count.

$$\sum_{k\ge 1} k\binom{m+1}{r+k+1} = \sum_{i=1}^{m} i2^{i-1}\binom{m-i}{r}$$

5. The line segments from $(j, \ln j)$ to $(j+1, \ln(j+1))$ lie below the curve $y = \ln x$ (since $f(x) = \ln x$ is convex). By comparing the area under the segments from $j = 1$ to $j = n$ with the area under the curve $y = \ln x$ from $x = 1$ to $x = n+1$, show that $n! \le e\sqrt{n+1}(n/e)^n$. [Hint: use that $1 + x \le e^x$ for real $x$.]

6. Flags on poles.

(a) Obtain a simple formula for the number of ways to put $m$ distinct flags on a row of $r$ flagpoles. Poles may be empty, and changing the order of flags on a pole changes the arrangement. The formula must only use one "$m$" and one "$r$". (The answer is 6 for $m = r = 2$, as shown below.)




(b) Prove that the identity below for rising factorials holds for all $x, y \in \mathbb{R}$.

$$(x+y)^{(n)} = \sum_k \binom{n}{k} x^{(k)} y^{(n-k)}$$