Name: Solutions

Directions: Show all work. No credit for answers without work. Except when otherwise directed, you may leave answers in terms of binomial/multinomial coefficients, factorials, and sums with a small number of terms.

1. **[7 points]** How many subsets of $\{1, \ldots, n\}$ have size 4? Give a simple formula in terms of n without using binomial/multinomial coefficients, factorials, or sums.

$$\binom{n}{4} = \frac{n!}{4!\,(n-4)!} = \frac{n_{(4)}}{4!} = \boxed{\frac{n(n-1)(n-2)(n-3)}{24}}$$

2. **[7 points]** For $n \geq 2$, what are the odds that a random permutation of $\{1, \ldots, n\}$ begins with two consecutive integers?

$\Omega = \{$all permutations of $[n]\}$, $|\Omega| = n!$

$A = \{\sigma \in \Omega : \sigma$ begins with consecutive integers$\}$, $|A| = \underbrace{(n-1)}_{\text{consecutive pair in first 2 slots}} \cdot \underbrace{2}_{\text{order pair chosen}} \cdot \underbrace{(n-2)!}_{\text{fill in remaining spots}}$

$$\Pr(A) = \frac{|A|}{|\Omega|} = \frac{(n-1)\cdot 2 \cdot (n-2)!}{n!} = \frac{\cancel{(n-1)}\cdot 2 \cdot \cancel{(n-2)!}}{n\cancel{(n-1)}\cancel{(n-2)!}} = \boxed{\frac{2}{n}}$$

3. **[2 parts, 6 points each]** A standard deck of cards has one card for each suit/rank pair, where the suits are spades, hearts, diamonds, and clubs, and the ranks are ace, 2 through 10, jack, queen, and king.

   (a) How many ways are there to choose a set of 6 cards with no spades?

   Choose 6 from the 39 non-spades: $\boxed{\binom{39}{6}}$

   (b) How many ways are there to choose a set of 6 cards which has at least one card in each of the four suits?

Case 1: 3 in one suit, one in each of the other 3:

1. Choose suit for 3 cards (4 opts)
2. Choose 3 cards from that suit ($\binom{13}{3}$ opts)
3. Choose 1 card from other 3 suits 13^3

Total: $4\binom{13}{3}13^3$

Case 2: 2 cards in two suits, 1 card in the other 2 suits

1. Choose suits with 2 cards ($\binom{4}{2}$ opts)
2. Choose 2 cards in those suits ($\binom{13}{2}^2$ opts)
3. Choose 2 cards in other two suits (13^2 opts)

Total: $\binom{4}{2}\binom{13}{2}^2 13^2$

By rule of sum, the total number is $\boxed{4\binom{13}{3}13^3 + \binom{4}{2}\binom{13}{2}^2 13^2}$.

4. **[3 parts, 5 points each]** How many ways are there to arrange the letters in the word KNICKKNACK:

   (a) without any restrictions?

   K N I C A
   K N C
   K
   K

   10 letters, 4 K's, 2N's, 2C's, 1I, 1A.

   So $\boxed{\binom{10}{4,2,2,1,1}}$ or $\frac{10!}{4!\,(2!)^2(1!)^2} = \boxed{\frac{10!}{4!(2!)^2}}$

   (b) that does not start or end with a 'K'?

   $- \underline{- - - - - - - -} -$

   ① Choose locations for the K's: $\binom{8}{4}$

   ② Fill in remaining 6 spaces: $\binom{6}{2,2,1,1}$

   $\boxed{\binom{8}{4}\cdot\binom{6}{2,2,1,1}}$ or $\frac{8!}{4!4!}\cdot\frac{6!}{(2!)^2}$

   $\frac{8\cdot7\cdot6\cdot5\cdot6\cdot5}{2^2}$ or $14\cdot(30)^2$ or $\boxed{12600}$

   (c) where the substring 'CKC' appears?

   Since CKC can only appear once, this is equivalent to arrangements of <CKC>KKKNNIA and there are $\boxed{\binom{8}{3,2,1,1,1}}$ or $\boxed{\frac{8!}{3!\,2!}}$ of these.

5. **[3 parts, 5 points each]** A donut shop sells 5 different kinds of donuts.

   (a) How many ways are there to purchase 24 donuts? (The order the donuts are purchased is unimportant.)

   24 stars, 4 bars (dividing the stars into 5 groups).

   So $\binom{24+4}{4} = \boxed{\binom{28}{4}}$

   (b) How many ways are there to purchase 24 donuts if we must have at least 3 of each type?

   Equivalent to # non-neg. integral solns to $x_1+\cdots+x_5=24$, each $x_i\geq 3$

   " " " " $x_1+\cdots+x_5=9$, each $x_i\geq 0$.

   9 stars, 4 bars $\Rightarrow \binom{9+4}{4} = \boxed{\binom{13}{4}}$

   (c) How many ways are there to purchase 24 donuts if we must purchase an even number of each type? (Note that zero is even, so not every type need be purchased.)

   Count the non-neg integral solns to $2x_1+2x_2+2x_3+2x_4+2x_5=24$,

   This is just # non-neg. integer solns to $x_1+\cdots+x_5=12$

   12 stars, 4 bars $\Rightarrow \boxed{\binom{16}{4}}$

6. **[2 parts, 10 points each]** Give combinatorial proofs for each of the following.

(a) $\binom{n}{k}\binom{k}{2} = \binom{n}{2}\binom{n-2}{k-2}$

Pf Both sides count the number of committees of size k that can be formed from a set of size n with 2 co-chairs.

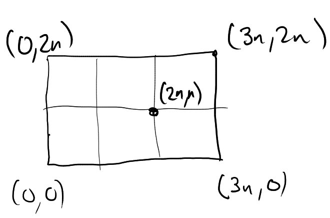
In the LHS, we first choose the committee members ($\binom{n}{k}$ options), and then we choose the co-chairs ($\binom{k}{2}$ options). In the RHS, we first choose the co-chairs ($\binom{n}{2}$ options) and then the remaining $k-2$ committee members ($\binom{n-2}{k-2}$ opts). $\blacksquare$

(b) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}2^k = 3^n.$

Pf Both sides count the lists of length n with entries in $\{0,1,2\}$. Let $A = \{0,1,2\}^n$, so that A is the set of such lists. We have $|A| = 3^n$ by the product principle.

For the LHS, we partition A into sets $A_0, \dots, A_n$ where A_k is the set of lists in A where k of the entries are in $\{0,1\}$ and $n-k$ of the entries are 2. Note that $|A_k| = \binom{n}{k}2^k$ by the product principle since we first choose the k positions to have a 0 or 1 ($\binom{n}{k}$ options), and then we fill each of those positions with a 0 or 1 (2^k options); the remaining positions must be filled with 2's. We have $3^n = |A| = \sum_{k=0}^{n}|A_k| = \sum_{k=0}^{n}\binom{n}{k}2^k$. $\blacksquare$

7. **[6 points]** How many lattice paths from $(0,0)$ to $(3n,2n)$ avoid the point $(2n,n)$? For example, when $n=1$, the answer is 4.



Count the complement/subtraction principle:

- All lattice paths: $\binom{2n+3n}{2n} = \binom{5n}{2n}$
- Lattice paths through $(2n,n)$: $\binom{n+2n}{n} \cdot \binom{n+n}{n}$

path from $(0,0)$ to $(2n,n)$ path from $(2n,n)$ to $(3n,2n)$

So total # of paths avoiding $(2n,n)$ is

$\boxed{\binom{5n}{2n} - \binom{3n}{n}\binom{2n}{n}}$.

8. **[2 parts, 6 points each]** A permutation σ of $\{1, \ldots, n\}$ is chosen at random.

   (a) What is the expected value of the first entry in σ?

Let X be the first entry.

We compute

$$E(X) = \sum_{k=1}^{n} k \cdot \Pr(X=k) = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{1}{n} \sum_{k=1}^{n} k = \frac{1}{n}\binom{n+1}{2}$$

$$= \frac{1}{\cancel{n}} \cdot \frac{(n+1)\cancel{n}}{2} = \boxed{\frac{n+1}{2}}.$$

   (b) What is the expected value of the sum of the first k entries in σ? (Hint: for $1 \le i \le n$, let $X_i = i$ if i appears in the first k positions of σ; otherwise let $X_i = 0$.)

Let X_i be as in the hint; let X be the sum of the first k positions, so that $X = \sum_{i=1}^{n} X_i$.

Note $E(X_i) = i \cdot \Pr(i \text{ in first } k \text{ positions}) + 0 \cdot \Pr(i \text{ not in first } k \text{ positions})$

$$= i \cdot \frac{k}{n} + 0 \cdot \frac{n-k}{n} = i \cdot \frac{k}{n}$$

We have $E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} i \frac{k}{n} = \frac{k}{n} \sum_{i=1}^{n} i = \frac{k}{n}\binom{n+1}{2} = \frac{k}{\cancel{n}} \frac{(n+1)(\cancel{n})}{2}$

$$= \boxed{\frac{k(n+1)}{2}}$$

9. **[6 points]** How many ways are there to form 2 groups of size 4 and 3 groups of size 5 from a class of 23 students?

Use division principle:

$$\binom{23}{4,4,5,5,5} \cdot \frac{1}{2!} \cdot \frac{1}{3!}$$

form ordered/distinguishable groups

divide by 2! ways to order the groups of size 4

divide by 3! ways to order the 3 groups of size 5

$$= \boxed{\binom{23}{4,4,5,5,5} \cdot \frac{1}{2!} \cdot \frac{1}{3!}} = \boxed{\frac{23!}{(4!)^2(5!)^3(2!)(3!)}}$$