Name: ____________________________

Directions: Show all work. No credit for answers without work.

1. Prove the following by induction or the no minimum counter-example technique.

   (a) **[15 points]** If $n \geq 0$, then $\displaystyle\sum_{k=1}^{n}(k^2 + 2k - 1)2^{k-1} = n^2 2^n$.

   (b) **[15 points]** If $n \geq 0$, then $n < 2^n$.

2. **[20 points]** A store sells 11-cent and 20-cent stamps. Prove that for each integer n such that $n \geq 190$, it is possible to purchase some combination of stamps with total postage n cents. (Hints: consider a proof by induction on n with basis step $n = 190$. It may be helpful to note that $(5)(20) + (-9)(11) = 1$ and $(-6)(20) + (11)(11) = 1$.)

3. **[2 parts, 15 points each]** For $n \geq 0$, let A_n be the set of all lists of length n with entries in $\{0, 1, 2\}$ not containing consecutive 1's or consecutive 2's. For example, $A_2 = \{00, 01, 02, 10, 12, 20, 21\}$ and $|A_2| = 7$. Let $a_n = |A_n|$.

   (a) For $n \geq 1$, let B_n be the lists in A_n ending in 0, and let C_n be the lists in A_n ending in 1 or 2. Prove that $|A_n| = 3|B_{n-1}| + 2|C_{n-1}|$ for $n \geq 2$.

   (b) Use part (a) to obtain a second-order linear homogeneous recurrence for a_n. Be sure to include base cases. Do not solve the recurrence.

4. **[20 points]** Consider a party with $2k$ people, where $k \geq 2$. Each pair of people shakes hands at most once, and every person shakes hands with an even number of people. Show that there are three people at the party who shake hands the same number of times.