Name: Solutions

Directions: Show all work. You may leave your answer in terms of factorials, falling factorials, and binomial coefficients. Any sums or products should be as simple as possible.

1. **[1 point]** How many ways can a list of 4 distinct integers be formed from $\{1, \ldots, 10\}$ if the first two integers must sum to 8?

$\overset{5}{4}8$
$\times\ 7$
336

First 2: 1 7, 2 6, 3 5, 5 3, 6 2, 7 1 (6 options) Note: 4 4 not allowed.

Third integer: From $\{1, \ldots, 10\}$ distinct from first two (8 options)

Fourth integer: From $\{1, \ldots, 10\}$ distinct from first three (7 options)

So # ways $= 6 \cdot 8 \cdot 7 = 8_{(3)} = \boxed{336}$

2. **[2 parts, 3 points each]** A class of 9 students needs to be split into 3 groups of 3 students.

(a) How many ways can this be done, with no additional restrictions?

With ordered groups: $\binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3}$

To make the groups unordered/indistinguishable: $\frac{1}{3!}\binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3} = \frac{1}{3!} \cdot \frac{9!}{\cancel{6!}\,3!} \cdot \frac{\cancel{6!}}{3!\cdot 3!} \cdot 1$

$= \frac{\cancel{9}^{3} \cdot \cancel{8}^{2} \cdot 7 \cdot \cancel{6} \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{3 \cdot 2} \cdot \cancel{3 \cdot 2} \cdot \cancel{3 \cdot 2} \cdot \cancel{3 \cdot 2}} = 2 \cdot 7 \cdot 20 = \boxed{280}$

(b) Suppose that two particular students, say students 8 and 9, do not get along and cannot be assigned to the same group. Now how many ways are there to assign the students to groups?

① Choose pair with 9: ($\binom{7}{2}$ options)

② Choose pair with 8: ($\binom{5}{2}$ options)

Last Group is forced.

Total: $\binom{7}{2} \cdot \binom{5}{2} = \frac{7 \cdot 6}{2} \cdot \frac{5 \cdot 4}{2} = 7 \cdot 3 \cdot 10 = \boxed{210}$

3. **[3 points]** A standard deck of cards has 4 suits (spades, hearts, diamonds, and clubs), and 13 ranks (ace, 2 through 10, jack, queen, and king), and contains one card for each suit/rank pair. Suppose that all 52 cards are ordered randomly. What are the odds that every spade appears before every heart?

Soln 1: Imagine 52 spaces.

① Choose a set S of 26 spaces for spades/hearts $\binom{52}{26}$ options

② Order spades in first 13 spaces of S: $13!$ opts

③ order hearts in last 13 spaces of S: $13!$ opts

④ Order other cards in remaining spaces: $26!$ opts.

Total # orderings with all spades before hearts: $\frac{52!}{26! \cdot \cancel{26!}} \cdot 13! \cdot 13! \cdot \cancel{26!}$

Probability is $\frac{\frac{\cancel{52!}}{26!}(13!)(13!)}{\cancel{52!}}$

$= \frac{(13)! \cdot (13)!}{(26!)} = \boxed{\frac{1}{\binom{26}{13}}}$