Name: Solutions

Directions: Show all work.

1. **[4 points]** Give a list of distinct integers of maximum size which has no increasing subsequence of size 3 and no decreasing subsequence of size 6. (You do not need to prove your list has maximum size.)

$a+1=3$, $b+1=6$; lists of size $\geq a\cdot b+1$ are too large. Want $n=ab=2\cdot 5=10$.

L: $\boxed{9, 10, 7, 8, 5, 6, 3, 4, 1, 2}$

2. For $n \geq 3$, let $a_n = -a_{n-1} + 8a_{n-2} + 12a_{n-3}$.

(a) **[3 points]** Find the general solution to the recurrence.

Char Eqn $x^3 + x^2 - 8x - 12 = 0$

$x=-2$ is a root

$$\begin{array}{r|l} & x^2 - x - 6 \\ x+2 & x^3 + x^2 - 8x - 12 \\ & x^3 + 2x^2 \\ \hline & -x^2 - 8x - 12 \\ & -x^2 - 2x \\ \hline & -6x - 12 \end{array}$$

$x^3 + x^2 - 8x - 12 = 0$

$(x+2)(x^2 - x - 6)$

$(x+2)(x+2)(x-3) = 0$

$(x+2)^2(x-3) = 0$

$x=-2$ mult 2 $\quad$ $x=3$ mult 1

So Gen soln:

$$\boxed{a_n = (An+B)(-2)^n + C(3)^n}$$

(b) **[3 points]** Find the solution when $(a_0, a_1, a_2) = (0, 4, 9)$.

$a_n = -a_{n-1} + 8a_{n-2} + 12a_{n-3}$

$a_0 = 0$: $0 = (A\cdot 0 + B)1 + C = B + C$

$a_1 = 4$: $4 = (A\cdot 1 + B)(-2)^1 + C\cdot 3 = -2A - 2B + 3C$

$a_2 = 9$: $9 = (2A + B)(-2)^2 + C\cdot 9 = 8A + 4B + 9C$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ -2 & -2 & 3 & 4 \\ 8 & 4 & 9 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ -2 & -2 & 3 & 4 \\ 0 & -4 & 21 & 25 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & 2 & -3 & -4 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & 21 & 25 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 2 & -3 & -4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 25 & 25 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & 2 & -3 & -4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 0 & 0 & +1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & +\frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So $A = \frac{1}{2}$, $B = -1$, $C = 1$

and

$$\boxed{a_n = \left(\tfrac{1}{2}n - 1\right)(-2)^n + 3^n}$$

Check:

$\cancel{-23} + 72 + \cancel{48}$ $\;25$

n	0	1	2	3	4
a_n	0	4	9	23 ✓	97 ✓