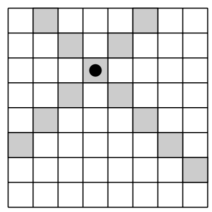
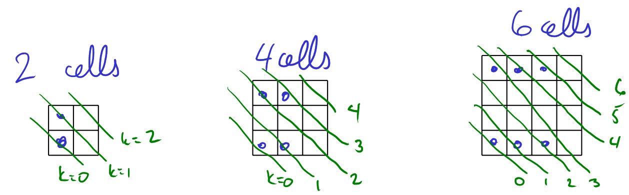
Name: Solutions

Directions: Show all work.

1. Consider the $(n \times n)$-grid, with set of cells $\{(x, y)\colon\ 0 \le x \le n-1 \text{ and } 0 \le y \le n-1\}$. Two cells (x, y) and (x', y') *share a diagonal* if $y + x = y' + x'$ or $y - x = y' - x'$. Note that when $y + x = y' + x'$, both cells are on the same diagonal with slope -1 and when $y - x = y' - x'$, both cells are on the same diagonal with slope 1. For example, the cells that share a diagonal with $(3, 5)$ in the (8×8)-grid are shaded below left.





(a) **[3 points]** For $n \in \{2, 3, 4\}$, give a maximum-sized set of cells with no two cells sharing a diagonal in the grids above right. (No need to prove your answer is correct.)

(b) **[2 points]** For $n \ge 2$, make a conjecture for the maximum size of a set of cells in the $(n \times n)$-grid with no two cells sharing a diagonal.

Conj. The maximum size of such a set is $2n-2$.

(c) **[5 points]** Prove that your conjecture is correct.

Pf. To see that the max. size is at least $2n-2$, we give a construction. Let $S = \{(x,y) : 0 \le x \le n-2 \text{ and } y \in \{0, n-1\}\}$. Note that $|S| = 2n-2$. Also, the sums $x+y$ for $(x,y) \in S$ are $0, 1, \dots, n-2, n-1, \dots, (n-1)+(n-2)$ and so they are all distinct. Also the differences $y-x$ for $(x,y) \in S$ are $0, -1, \dots, -(n-2), (n-1), \dots, (n-1)-(n-2)$, so they are distinct also. So no two cells in S share a diagonal.

To see that the max size is at most $2n-2$, let S be a set of cells in the $(n \times n)$-grid with no two sharing a diagonal. For $0 \le k \le 2n-2$, let D_k be the set of cells (x,y) such that $x+y = k$. Since all cells in D_k are on a common diagonal, S contains at most 1 cell in D_k. Since k ranges from 0 to $2n-2$, it follows that $|S| \le 2n-1$. For $|S| = 2n-1$, it must be that S has one cell from each diagonal $D_0, D_1, \dots, D_{2n-2}$. In particular, since $D_0 = \{(0,0)\}$ and $D_{2n-2} = \{(n-1, n-1)\}$, it follows that S has both $(0,0)$ and $(n-1, n-1)$. But these points are on the same diagonal, so this is impossible. Therefore $|S| \le 2n-2$. $\blacksquare$