Name: Solutions

Directions: Show all work.

1. **[5 points]** Use the binomial theorem to find a closed-form formula for $\sum_{k=0}^{n} k2^k\binom{n}{k}$.

The binomial theorem states $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. Setting $y=1$ gives $(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k(1)^{n-k}$ or $(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$. Differentiating the LHS gives $\frac{d}{dx}\left[(x+1)^n\right] = n(x+1)^{n-1}\frac{d}{dx}[x+1] = n(x+1)^{n-1}$. Differentiating the RHS gives $\frac{d}{dx}\left[\sum_{k=0}^{n}\binom{n}{k}x^k\right] = \sum_{k=0}^{n}\frac{d}{dx}\left[\binom{n}{k}x^k\right] = \sum_{k=0}^{n}k\binom{n}{k}x^{k-1}$. Therefore

$$n(x+1)^{n-1} = \sum_{k=0}^{n}k\binom{n}{k}x^{k-1}.$$

Setting $x=2$ gives $\sum_{k=0}^{n}k\binom{n}{k}2^{k-1} = n(3)^{n-1}$ and finally multiplying both sides by 2 gives

$$\sum_{k=0}^{n}k\binom{n}{k}2^k = \boxed{2n3^{n-1}}.$$

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2. **[5 points]** A party has a total of $3n$ people, with n single people and $2n$ people in n couples. Find a summation formula for the number of ways to select n people such that each couple has at most 1 selected person.

Solution 1: Inclusion/Exclusion. Let $Z=\{z_1,\dots,z_n\}$ and $X=\{x_1,\dots,x_n,y_1,\dots,y_n\}$; we view Z as the set of single people and X as the set of couples, where the i^{th} couple is $\{x_i,y_i\}$. Let $\mathcal{U}=\binom{X\cup Z}{n}$ so that $\mathcal{U}$ is the set of all n-subsets of $X\cup Z$. For each i, let A_i be the "bad set" consisting of all the n-subsets of $X\cup Z$ where both x_i and y_i are selected. We seek $\left|\overline{\bigcup_{i=1}^{n}A_i}\right|$. Note that if $S\subseteq[n]$ and $|S|=k$, then $\left|\bigcap_{i\in S}A_i\right| = \binom{3n-2k}{n-2k}$. Therefore $\left|\overline{\bigcup_{i=1}^{n}A_i}\right| = \boxed{\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{3n-2k}{n-2k}}$.

Solution 2: Summation Principle. Let Z and X be as in Soln 1. Let A be the set of all n-subsets of $X\cup Z$ that contain at most 1 person from each couple $\{x_i,y_i\}$. Let A_k be the set of all n-sets $Q\in A$ such that $|Q\cap X|=k$ and $|Q\cap Z|=n-k$. Note that $|A_k|=\binom{n}{k}2^k\binom{n}{n-k}$, since we obtain a set $Q\in A_k$ by choosing k couples ($\binom{n}{k}$ options), choosing x_i or y_i for each chosen couple (2^k options) and finally choosing $n-k$ people from the set Z of single people. Hence $|A| = \sum_{k=0}^{n}|A_k| = \sum_{k=0}^{n}\binom{n}{k}2^k\binom{n}{k} = \boxed{\sum_{k=0}^{n}2^k\binom{n}{k}^2}$.

Note: Since both solutions count the same set, we have $\boxed{\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{3n-2k}{n-2k} = \sum_{k=0}^{n}2^k\binom{n}{k}^2}$.