Name: Solutions

Directions: Show all work.

1. **[5 points]** Let $n \geq 1$. Give the identity that is proved by the following combinatorial argument: Each non-empty subset of $\{1, \ldots, n\}$ consists of a maximum element k for some $1 \leq k \leq n$ together with a subset of $\{1, \ldots, k-1\}$.

The number of non-empty subsets of $[n]$ is $2^n - 1$

The number of subsets of $\{1, \ldots, k-1\}$ is 2^{k-1}. Therefore the argument proves

$$\boxed{\sum_{k=1}^{n} 2^{k-1} = 2^n - 1.}$$

2. **[5 points]** Let $n \geq 1$. Give a combinatorial proof that $n! - 1 = \sum_{k=1}^{n} (k-1)(k-1)!$. (Hint: except for the identity permutation σ where $\sigma = 12 \cdots n$, every permutation σ of $[n]$ has at least one index i such that the value of σ at position i is not equal to i.)

Let A be the set of permutations σ of $[n]$ such that σ is not the identity. We count A in two ways.

Since there are $n!$ permutations of $[n]$ and only one of them is the identity we have $|A| = n! - 1$, giving the LHS.

For the RHS, let A_k be the set of permutations σ in A such that k is the max integer with $\sigma(k) \neq k$. As noted in the hint, every non-identity permutation has at least one index i such that $\sigma(i) \neq i$, and so $\{A_1, \ldots, A_n\}$ is a partition of A, implying $|A| = \sum_{k=1}^{n} |A_k|$. To count A_k, we note that each $\sigma \in A_k$ satisfies $\sigma(j) = j$ for $j > k$ and so σ is determined by the permutation of the elements $\{1, \ldots, k\}$ in the first k positions of σ. Since $\sigma(k) \neq k$, it must be that $\sigma(k) \in \{1, \ldots, k-1\}$, and so there are $k-1$ options for $\sigma(k)$. We complete σ by arranging the remaining symbols in $\{1, \ldots, k\} - \{\sigma(k)\}$ in the first $k-1$ positions of σ. So $|A_k| = (k-1)(k-1)!$ by the product principle.

Therefore $n! - 1 = |A| = \sum_{k=1}^{n} |A_k| = \sum_{k=1}^{n} (k-1)(k-1)!$. ▨