Name: Solutions

Directions: Show all work.

1. **[5 points]** Prove for $n \geq 0$, we have $\sum_{k=1}^{n} k(k+2) = \frac{1}{6}[n(n+1)(2n+7)]$.

Pf: By induction on n.

Basis Step: If $n=0$, then the LHS is the empty sum, and hence 0, and the RHS is $\frac{1}{6}[0 \cdot 1 \cdot 7] = 0$.

Inductive Step: Suppose $n \geq 1$. By the inductive hypothesis, we have $\sum_{k=1}^{n-1} k(k+2) = \frac{1}{6}[(n-1)n(2(n-1)+7)]$.

Adding $n(n+2)$ to both sides gives

$$\sum_{k=1}^{n} k(k+2) = \frac{1}{6}[(n-1)n(2n+5)] + n(n+2) = \frac{1}{6}[(n-1)n(2n+5) + 6n(n+2)]$$

$$= \frac{1}{6}[n[(n-1)(2n+5) + 6(n+2)]] = \frac{1}{6}[n[2n^2 - 2n + 5n - 5 + 6n + 12]]$$

$$= \frac{1}{6}[n[2n^2 + 9n + 7]] = \frac{1}{6}[n(n+1)(2n+7)].$$

Hence the identity holds at n. ∎

2. **[5 points]** Let t be a real number, let $a_0 = 0$, let $a_1 = t$, and let $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$. Prove that for $n \geq 0$, we have that $a_n = t(2^n - 1)$.

Pf. By induction on n.

Basis Step: If $n=0$, then $a_0 = 0$ and $t(2^n - 1) = t(2^0 - 1) = t(1-1) = 0$.

If $n=1$, then $a_1 = t$ and $t(2^n - 1) = t(2^1 - 1) = t(2-1) = t$.

Hence $a_n = t(2^n - 1)$ when $n \in \{0, 1\}$.

Inductive Step: Suppose $n \geq 2$. By the inductive hypothesis, we have $a_{n-1} = t(2^{n-1} - 1)$ and $a_{n-2} = t(2^{n-2} - 1)$. Since $n \geq 2$, the recurrence applies and we compute

$$a_n = 3a_{n-1} - 2a_{n-2} = 3(t(2^{n-1} - 1)) - 2(t(2^{n-2} - 1))$$

$$= 3t\,2^{n-1} - 3t - 2t\,2^{n-2} + 2t$$

$$= 6t\,2^{n-2} - 2t\,2^{n-2} - t$$

$$= 4t\,2^{n-2} - t$$

$$= t\,2^n - t = t(2^n - 1).$$

Hence the identity holds at n. ∎