Directions: Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. Let $a_0 = 4$, $a_1 = 10$, $a_2 = 88$, and $a_n = a_{n-1} + 21a_{n-2} - 45a_{n-3}$ for $n \geq 3$. Use the characteristic equation method to solve the recurrence.

2. For positive m and n, a domino tiling of an $m \times n$ grid is *rigid* if every horizontal and vertical cut crosses a domino. In this problem, we characterize the grids with even dimensions that have rigid domino tilings.

   (a) Prove that if $m = 2r$, $n = 2s$, and the $m \times n$ grid has a rigid domino tiling, then $(r-2)(s-2) \geq 2$. (Hint: generalize our argument in class that the 6×6 grid has no rigid domino tiling.)

   (b) Construct a rigid domino tiling of the 6×8 grid.

   (c) Prove that if $m \geq 3$ and the $m \times n$ grid has a rigid domino tiling, then the $m \times (n+2)$ grid also has a rigid domino tiling. (Hint: show how to modify an arbitrary rigid domino tiling of the $m \times n$ grid to obtain a rigid domino tiling of the $m \times (n+2)$ grid.)

   (d) Let m and n be positive, even integers with $m \leq n$. Prove that the $m \times n$ grid has a rigid domino tiling if and only if $m \geq 6$ and $n \geq 8$. (Hint: for the forward direction, use part (a) to give a direct proof. For the backward direction, use parts (b) and (c) to give an inductive proof.)

3. [2.1.{25,26}] Recall that the *slope* of the line segment joining the pair of points (x_1, y_1) and (x_2, y_2) in the plane is $(y_2 - y_1)/(x_2 - x_1)$.

   (a) Prove that if S as a set of 17 points in the plane, no two of which are on a common vertical or horizontal line, then there exist $p_1, \ldots, p_5 \in S$ such that the slope of the line segments joining p_i and p_j for $1 \leq i < j \leq 5$ all have the same sign.

   (b) Give an example that shows that the conclusion of part (a) does not always hold if we assume only that $|S| \geq 16$.