Directions: Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. For $b \geq 0$, let b_n be the number of ways to tile a $3 \times n$ grid with 1×3 rectangular tiles. Note that $b_0 = 1$, since placing zero tiles counts as a tiling of the 3×0 grid.

   (a) Find a recurrence relation for b_n. (Your recurrence should include all needed base cases.)

   (b) Recall that the number of ways a_n of tiling a $2 \times n$ grid with dominos is given by the recurrence $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. How does b_n compare with a_n, the number of ways to tile a $2 \times n$ grid with dominos? Explain. Can you prove your claim?

2. [SS 1.3.{8,9}] You work at a car dealership that sells three models: A pickup trick, an SUV, and a compact hybrid. Your job is to park the vehicles in a row. The pickup trucks and the SUVs take up two spaces while the hybrid takes up one space. Let n be a nonnegative integer and let $f(n)$ be the number of ways of arranging vehicles in exactly n spaces, with each space occupied.

   (a) Find a recurrence relation for $f(n)$ and use it to compute $f(0)$ through $f(10)$.

   (b) Find a first-order recurrence relation g that appears to match f (i.e. $g(n)$ should depends only on $g(n-1)$).

   (c) Prove that $g(n) = f(n)$ by induction.

   (d) Use the values for $f(0)$ and $f(1)$ to find a candidate formula for $f(n)$ of the form $f(n) = a2^n + b(-1)^n$. Prove that your formula is correct.

3. [2.1.6] You have a 3×3-square, and you throw 10 darts at it. Show that no matter where the darts land, there are two darts whose distance is at most $\sqrt{2}$.

4. [2.1.22] I have 51 rectangular pieces of cardboard, each of which has an integer length and width in the set $\{1, \ldots, 100\}$. (Note that squares are allowed.) Prove that there are two rectangles such that one can fully cover the other when placed on top.