Directions: Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. The *k-dimensional hypercube* Q_k has vertex set $\{0,1\}^k$ (all bitstrings of length k) with u and v adjacent if and only if u and v differ in one position. Prove that Q_k is bipartite.

2. [10.3.10] Let G be a multigraph with n vertices and m edges. Show that G has a bipartite subgraph that has n vertices and at least $m/2$ edges.

3. Let G be a k-regular bipartite graph. Prove that $E(G)$ can be partitioned into perfect matchings. (Hint: use Hall's Theorem to show that G has a perfect matching and then use induction on k.)

4. [10.7.4] Let G be a graph. Prove that $\chi(G) \leq 2$ if and only if G is bipartite.

5. [10.7.5] Let G be an n-vertex graph. Prove that $\alpha(G) \cdot \chi(G) \geq n$.

6. Chromatic number and odd cycles.

   (a) Let G be a graph such that $\chi(G) \geq 9$. Prove that G has three vertex-disjoint odd cycles.

   (b) Find an example of a graph G with $\chi(G) = 8$ such that G does not contain three vertex-disjoint odd cycles.