**Directions:** Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

- 1. Prove that for every nonnegative integer n, we have  $\sum_{k=1}^{n} k^2 = [(2n+1)(n+1)n]/6$ .
- 2. Recall that  $n! = 1 \cdot 2 \cdots n$ . For  $n \ge 0$ , find a formula for  $1 + \sum_{k=1}^{n} k \cdot k!$  and prove your formula is correct.
- 3. The adjusted Fibonacci sequence  $\hat{F}_n$  is given by  $\hat{F}_0 = \hat{F}_1 = 1$  and  $\hat{F}_n = \hat{F}_{n-1} + \hat{F}_{n-2}$  for  $n \ge 2$ . (Note that  $\hat{F}_2 = \hat{F}_1 + \hat{F}_0 = 1 + 1 = 2$ , and  $\hat{F}_3 = \hat{F}_2 + \hat{F}_1 = 2 + 1 = 3$ .) Prove that for  $n \ge 0$ , we have  $\hat{F}_n \le \phi^n$ , where  $\phi = (1 + \sqrt{5})/2$ .
- 4. A unit fraction is a rational number of the form 1/n for some positive integer n. An Egyptian fraction is the sum of zero or more distinct unit fractions. For example,  $\frac{29}{45}$  is an Egyptian fraction since  $\frac{29}{45} = \frac{1}{4} + \frac{1}{5} + \frac{1}{9} + \frac{1}{12}$ . Although  $\frac{11}{12} = \frac{1}{3} + \frac{1}{3} + \frac{1}{4}$ , this is not enough to establish that  $\frac{11}{12}$  is an Egyptian fraction since the unit sums are not distinct. However,  $\frac{11}{12} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6}$  does establish that  $\frac{11}{12}$  is an Egyptian fraction.
  - (a) Let a and b be nonnegative integers with 0 < a < b, and let n be the smallest positive integer such that  $\frac{1}{n} \leq \frac{a}{b}$ . Prove that  $\frac{a}{b} \frac{1}{n} = \frac{c}{d}$  for some nonnegative integers c and d such that c < a and  $\frac{c}{d} < \frac{1}{n}$ .
  - (b) Prove that if a and b are nonnegative integers with a < b, then  $\frac{a}{b}$  is an Egyptian fraction.

Comment: with part (b) and the fact that the Harmonic series  $1 + \frac{1}{2} + \frac{1}{3} + \cdots$  diverges, one can show that every nonnegative rational number is an Egyptian fraction.