Name: ____________________

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[10 points]** Give a contrapositive proof for the following. Suppose $z \in \mathbb{R}$. If $z \neq 1$ and $z \neq 4$, then $z^2 + 4 \neq 5z$.

2. **[10 points]** Let $a, b, a', b' \in \mathbb{Z}$ and let $m \in \mathbb{N}$. Show that if $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$, then $a + b \equiv a' + b' \pmod{m}$.

3. **[10 points]** Let $x \in \mathbb{R}$. Give a proof by contradiction that x^2 is rational or $(\sqrt{2}) \cdot x$ is irrational.

4. **[2 parts, 10 points each]** Powers of three.

   (a) Let $a, c \in \mathbb{Z}$. Prove that if $3^a < 3^c$, then $\frac{3^a}{3^c} \leq \frac{1}{3}$. (You may use the fact that $f(x) = 3^x$ is an increasing function.)

   (b) Use part (a) to show that for all $a, b, c \in \mathbb{Z}$, we have $3^a + 3^b \neq 3^c$.

5. **[5 points]** What is the coefficient of x^5y^6 in the expansion of $(x+y)^{11}$? Give a simplified, numerical answer.

6. **[2 parts, 10 points each]** Algebraic and Combinatorial Proofs. Let $k, n \in \mathbb{Z}$ with $0 \leq k \leq n$.

   (a) Give an algebraic proof that $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$.

   (b) Give a combinatorial proof of the same identity. (Hints: let $U = \{1, \dots, n\}$. Color k of the integers in U red and the other $n-k$ integers blue. Partition the 2-subsets of U into three groups.)

7. **[15 points]** Let $a, b \in \mathbb{Z}$. Show that $b \mid a$ and $b \mid a+1$ if and only if $b = -1$ or $b = 1$.

8. **[10 points]** Suppose $a, b, c, d \in \mathbb{R}$. Prove that if $a \neq c$ or $b \neq d$, then there is at most one $x \in \mathbb{R}$ such that $ax + b = cx + d$.