Name: ______________________________

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[6 parts, 4 points each]** First, express the following statements using the standard logical operators $\vee, \wedge, \sim, \Rightarrow, \Leftrightarrow$ and given open sentences. Second, state whether the statement is true or false (write the entire word); no justification necessary.

$P_1 : 2 + 3 = 8$ $\quad$ P_2 : red is a color $\quad$ $Q(X)$: X is an infinte set

$R(x)$: x is prime $\quad$ $S(x)$: x is odd

(a) Red is a color and $2 + 3 = 8$.

(b) $2 + 3 \neq 8$.

(c) If 3 is not odd, then red is not a color.

(d) The integer 5 is odd if and only if $\mathbb{Z}$ is an infinite set.

(e) For 23 to be prime, it is sufficient that 8 is odd.

(f) Either 21 is prime or 9 is odd, but not both.

2. **[2 points]** What 1900-era discovery prompted an overhaul of formal mathematics, and why?

3. Truth table

   (a) **[6 points]** Give a truth table for $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$.

   (b) **[4 points]** Find a simple formula which is equivalent to $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$.

4. Let ϕ_1 be the formula $(P \Rightarrow Q) \Rightarrow R$ and let ϕ_2 be the formula $P \Rightarrow (Q \Rightarrow R)$.

   (a) **[6 points]** Find a setting of truth values for P, Q, and R that makes ϕ_1 false and ϕ_2 true.

   (b) **[4 points]** Based on part (a), what can we conclude about ϕ_1 and ϕ_2?

5. **[5 points]** Consider the following definition: "An integer n is *large* if it takes more than 20 seconds to write down n in decimal." What is problematic about this definition? How can those problems be addressed?

6. **[4 parts, 4 points each]** First, translate the following statements in formal logic to English as naturally as possible. Second, state whether the statement true or false (write the entire word); give brief justifications where appropriate for partial credit.

   (a) $\exists n \in \mathbb{Z}, (\exists s \in \mathbb{Z}, n = 2s) \wedge (\exists t \in \mathbb{Z}, n = 2t + 1)$

   (b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y < 0$

   (c) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y < 0$

   (d) $\forall m \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq m \wedge (\forall k \in \mathbb{N}, 1 < k < n \Rightarrow \frac{n}{k} \notin \mathbb{N})$

7. **[3 parts, 3 points each]** Negate each sentence below in English as naturally as possible.

   (a) Some integer is both a perfect square and a perfect cube.

   (b) Every set of real numbers has a positive real number as a member.

   (c) For each nonempty set A of real numbers, if $a \leq 100$ for each $a \in A$, then there exists $M \in A$ such that $b \leq M$ for each $b \in A$.

8. **[8 points]** Prove that if n is an odd integer, then $n^2 - 1$ is a multiple of 4.

9. **[2 parts, 8 points each]** A two-step proof. In both parts, let a, b, and d be integers.

   (a) Prove that if $d \mid b$ and $d \mid a + b$, then $d \mid a$.

   (b) Use part (a) to show that if $d \mid an + b$ for each $n \in \mathbb{Z}$, then $d \mid a$ and $d \mid b$.