Math 303 **Quiz 11** **Fri. Apr. 12, 2024**

Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[3 points]** Let $a \in \mathbb{R}$. Prove that if a^3 is irrational, then a is irrational.

We prove the contrapositive: if a is rational, then a^3 is rational.
Suppose a is rational. This means that $a = \frac{r}{s}$ for some $r, s \in \mathbb{Z}$ with $s \neq 0$. It follows that $a^3 = \left(\frac{r}{s}\right)^3 = \frac{r^3}{s^3}$. Since $r^3, s^3 \in \mathbb{Z}$, it follows that a^3 is rational. $\blacksquare$

2. **[3 points]** Let $a, b \in \mathbb{Z}$ with $b > 0$. Prove that there is at most one pair of integers (q, r) such that $a = bq + r$ and $0 \leq r < b$.

Suppose that (q_1, r_1) and (q_2, r_2) satisfy the given conditions, namely that $a = bq_1 + r_1$ and $a = bq_2 + r_2$ with $0 \leq r_1, r_2 < b$.
We show that $q_1 = q_2$ and $r_1 = r_2$, which implies that at most one such pair of integers exist. Note that $bq_1 + r_1 = a = bq_2 + r_2$, and it follows that $b(q_1 - q_2) = r_2 - r_1$. Since $q_1 - q_2 \in \mathbb{Z}$, we have that $b \mid r_2 - r_1$, or equivalently, $r_2 - r_1$ is a multiple of b. Since $r_2 \leq b-1$ and $r_1 \geq 0$, we have $r_2 - r_1 \leq b-1$. Also, since $r_2 \geq 0$ and $r_1 \leq b-1$, we have $r_2 - r_1 \geq -(b-1)$.
So $r_2 - r_1$ is a multiple of b in the set $\{-(b-1), \ldots, 0, \ldots, b+1\}$. We conclude that $r_2 - r_1 = 0$, and so $r_1 = r_2$. From $b(q_1 - q_2) = r_2 - r_1 = 0$, we may divide by b since $b \neq 0$ to obtain $q_1 - q_2 = 0$. It follows that $q_1 = q_2$. $\blacksquare$

3. Let $a, b, c, d \in \mathbb{R}$, let $f(x) = ax + b$, and let $g(x) = cx + d$.

   (a) **[3 points]** Show that there exists $x \in \mathbb{R}$ such that $f(x) = g(x)$ if and only if $a \neq c$ or $d = b$.

($\Rightarrow$) Suppose that there exists $x \in \mathbb{R}$ such that $f(x) = g(x)$. Choose $x_0 \in \mathbb{R}$ such that $ax_0 + b = cx_0 + d$. Rearranging gives $(a-c)x_0 = d-b$. If $a \neq c$, then the conclusion is satisfied. Otherwise $a = c$ and we have $d-b = (a-c)x_0 = 0 \cdot x_0 = 0$, which implies $b = d$. In both cases, $a \neq c$ or $b = d$.

($\Leftarrow$). Suppose $a \neq c$ or $d = b$. We show that $f(x) = g(x)$ for some $x \in \mathbb{R}$.

Note that $f(x) = g(x)$ is equivalent to $ax + b = cx + d$, which is equivalent to $(a-c)x = d-b$. We consider two cases.

<u>Case 1</u>: If $a \neq c$, then we have $(a-c)x = d-b$ when $x = \frac{d-b}{a-c}$. Hence $f(x) = g(x)$ when $x = \frac{d-b}{a-c}$, and it follows that $f(x) = g(x)$ for some $x \in \mathbb{R}$.

<u>Case 2</u>: If $d = b$, then we have $(a-c)x = d-b = 0$ when $x = 0$. So $f(x) = g(x)$ for some $x \in \mathbb{R}$.

In both cases, there exists $x \in \mathbb{R}$ such that $f(x) = g(x)$. $\blacksquare$

   (b) **[1 point]** Fill in the blank to make the following statement true: There exists a unique $x \in \mathbb{R}$ such that $f(x) = g(x)$ if and only if $\underline{\quad a \neq c \quad}$.

<u>Note</u>: If $a \neq c$, then Case 1 of ($\Leftarrow$) direction shows that $x = \frac{d-b}{a-c}$ is the unique soln. If $a = c$, then there are either infinitely many solns ($b = d$) or no solns ($b \neq d$).