Directions: You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. See "Guidelines and advice" on the course webpage for more information.

1. Proof critiques. Give a critique of each claimed proof below. A proof critique addresses the following questions: (1) Is the proof correct? (2) If correct, can the proof be improved in some way? (3) If incorrect, what is/are the error(s)? Can they be fixed, and if so, how?

   (a) **Theorem 1.** *If x and y are real numbers, then $\frac{x+y}{2} \geq \sqrt{xy}$.*

   Proof:

$$\begin{aligned} \frac{x+y}{2} &\geq \sqrt{xy} \\ x+y &\geq 2\sqrt{xy} \\ (x+y)^2 &\geq 4xy \\ x^2+2xy+y^2 &\geq 4xy \\ x^2-2xy+y^2 &\geq 0 \\ (x-y)^2 &\geq 0 \end{aligned}$$

   □

   (b) **Theorem 2.** *All real numbers are equal.*

   Proof: Let x and y be real numbers. Observe that

$$x^2 - y^2 = (x-y)(x+y) = x(x+y) - y(x+y).$$

   After rearranging terms, this becomes $x^2 - x(x+y) = y^2 - y(x+y)$. Adding $\frac{(x+y)^2}{4}$ to both sides gives $x^2 - x(x+y) + \frac{(x+y)^2}{4} = y^2 - y(x+y) + \frac{(x+y)^2}{4}$. Factoring both sides, we see that $(x - \frac{x+y}{2})^2 = (y - \frac{x+y}{2})^2$ and taking the square root gives $x - \frac{x+y}{2} = y - \frac{x+y}{2}$. Adding $\frac{x+y}{2}$ to both sides gives $x = y$. Since x and y were arbitrarily chosen real numbers, it follows that all real numbers are equal. □

   (c) **Theorem 3.** *If $n \in \mathbb{Z}$, then $n^2 = 3k$ or $n^2 = 3k+1$ for some $k \in \mathbb{Z}$.*

   Proof: Suppose that $n \in \mathbb{Z}$. By the division algorithm, it follows that $n = 3q + r$ for some integers q and r with $0 \leq r < 3$. Since r is an integer and $0 \leq r < 3$, it follows that $r \in \{0, 1, 2\}$. We consider three cases, depending on the value of r.

   Case 1: If $r = 0$, then $n^2 = (3q+0)^2 = 9q^2 = 3(3q^2)$, and so $n^2 = 3k$ when we set k equal to the integer $3q^2$.

   Case 2: If $r = 1$, then $n^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$, and so $n^2 = 3k + 1$ when we set k equal to the integer $3q^2 + 2q$.

   Case 3: If $r = 2$, then $n^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1$, and so $n^2 = 3k + 1$ when we set k equal to the integer $3q^2 + 4q + 1$.

   In all cases, we have that $n^2 = 3k$ or $n^2 = 3k + 1$ for some integer k. □

   (d) **Theorem 4.** *Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.*

Proof: Since $a \mid b$, we have that $b = ka$ for some integer k. Similarly, since $b \mid c$, it follows that $c = kb$ for some integer k. Therefore $c = kb = k(ka) = k^2a$. Since k^2 is an integer, it follows that $a \mid c$. □

2. Use the method of direct proof to prove the following.

   (a) Suppose a is an integer. If $7 \mid 4a$, then $7 \mid a$. Hint: consider the equation $a = 8a - 7a$.

   (b) Suppose a, b, and c are integers. If $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

   (c) If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$.

   (d) Every odd integer is a difference of two squares. (Example: $7 = 4^2 - 3^2$).