Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[9 parts, 2 points each]** Decide whether the following are true or false. Indicate your answer by writing the entire word. No justification required.

$$A = \{\varnothing\} \quad B = \{1,2,3\} \quad C = \{\{1,2\},\{3\}\} \quad D = \{\varnothing, \{1,2,3\}\} \quad E = \{\{2,1\},\{2,3\}\}$$

(a) $3 \in B$

TRUE

(b) $3 \in C$

FALSE ($\{3\} \in C$)

(c) $A \in D$

FALSE ($A \subseteq D$)

(d) $B \in D$

TRUE

(e) $2 \subseteq B$

FALSE (2 not a set)

(f) $B \subseteq C$

FALSE ($1 \in B$ but $1 \notin C$)

(g) $B \subseteq D$

FALSE ($1 \in B$ but $1 \notin D$)

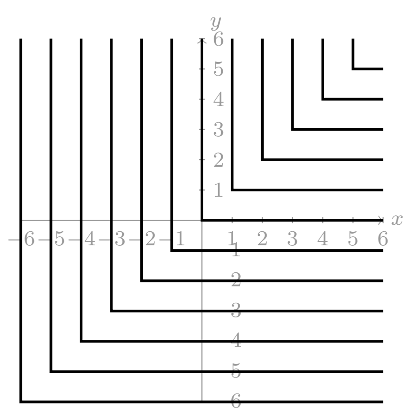
(h) $A \subseteq D$

True ($\varnothing \in D$ and $\varnothing$ is the only elt. in A)

(i) $C \subseteq E$

FALSE $\{3\} \in C$ but $\{3\} \notin E$.

2. **[6 points]** A subset of $\mathbb{R}^2$ is sketched below; the pattern in the figure below continues throughout the plane. Use set-builder notation to give a simple description of the set.



$$\boxed{\{(x,y) \in \mathbb{R}^2 : \min(x,y) \in \mathbb{Z}\}}$$

3. **[6 parts, 3 points each]** Express each set by listing the elements between braces.

$A = \{\{\}, \{1,2\}, \{2,1\}\}$ $B = \{\varnothing, 2, \{1,2\}, (1,2)\}$ $C = \{\varnothing, 1+1, \{2,1\}, (2,1)\}$ $D = \{\{1\}, 1, \{2\}, 2\}$

(a) $A \cap B$ $A = \{\emptyset, \{1,2\}\}$

$A \cap B = \boxed{\{\emptyset, \{1,2\}\}}$

(b) $B \cap C$

$B \cap C = \boxed{\{\emptyset, 2, \{1,2\}\}}$

(c) $(B \cup C) - A$

$\boxed{\{2, (1,2), (2,1)\}}$

(d) $(C - A) \times A$ $C - A = \{2, (2,1)\}$ $A = \{\emptyset, \{1,2\}\}$

$(C-A) \times A = \boxed{\{(2,\emptyset), (2,\{1,2\}), ((2,1),\emptyset), ((2,1),\{1,2\})\}}$

(e) $\mathcal{P}(B \cap D)$

$B \cap D = \{2\}$

$\mathcal{P}(B \cap D) = \boxed{\{\emptyset, \{2\}\}}$

(f) $(A \cup B \cup C) \cap \mathbb{Z}^2$

This set has all elts in A, B, and C that are ordered pairs of integers.

$\boxed{\{(1,2), (2,1)\}}$

4. **[3 parts, 4 points each]** Give an example or explain why no examples exist.

(a) A set A such that $(1,2) \in \mathcal{P}(A)$.

$\boxed{\text{Does not exist.}}$ Every element in $\mathcal{P}(A)$ is a set, and $(1,2)$ is not a set.

(b) Sets A and B such that $|A \times B| = 3$.

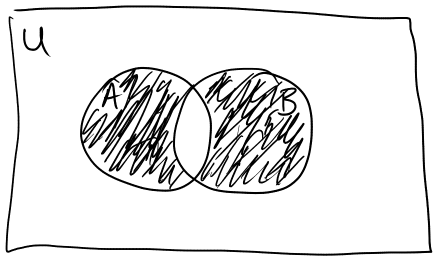
Any sets A and B with $|A| = 1$ and $|B| = 3$ will work. For example, $\boxed{A = \{1\} \text{ and } B = \{1,2,3\}}$.

(c) A nonempty set A such that $A \subseteq \mathcal{P}(A)$.

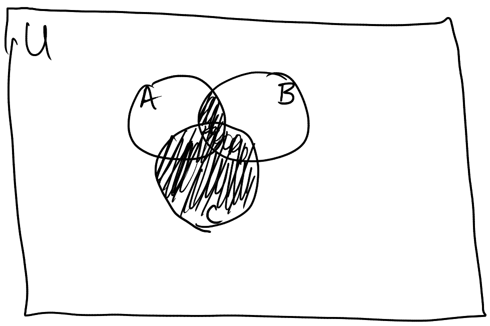
$A \subseteq \mathcal{P}(A)$ means every element in A is an elt in $\mathcal{P}(A)$ and hence $A \subseteq \mathcal{P}(A)$ means every elt in A is a subset of A. So take $\boxed{A = \{\emptyset\}}$.

5. **[4 parts, 3 points each]** Give Venn Diagrams for each of the following sets relative to a universe U.

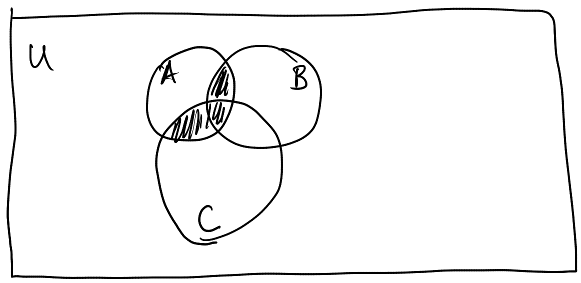
(a) $(A - B) \cup (B - A)$



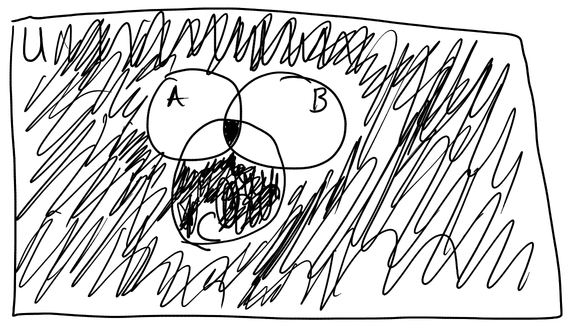
(b) $(A \cup C) \cap (B \cup C)$



(c) $(C \cup B) \cap A$



(d) $\overline{A \cup B} \cup (A \cap B \cap C)$



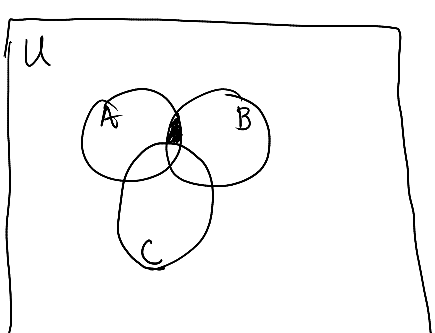
6. **[5 points]** Give two examples of an infinite set A such that $A \in \mathcal{P}(\mathcal{P}(\mathbb{Z}))$.

Many examples.

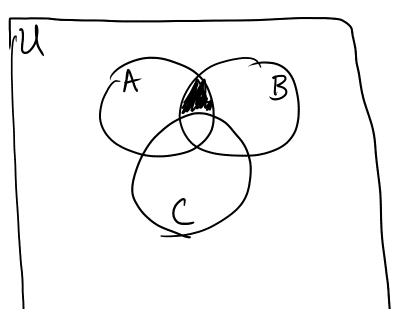
Ex 1: $A = \mathcal{P}(\mathbb{Z}) = \{X : X \subseteq \mathbb{Z}\}$

Ex 2: $A = \{X : X \subseteq \mathbb{Z} \text{ and } X \text{ is finite}\}$

7. **[5 points]** Use Venn Diagrams to decide if the equation $(A \cap B) - C = (A - C) \cap B$ is valid for all sets A, B, and C.



$(A \cap B) - C$



$(A - C) \cap B$

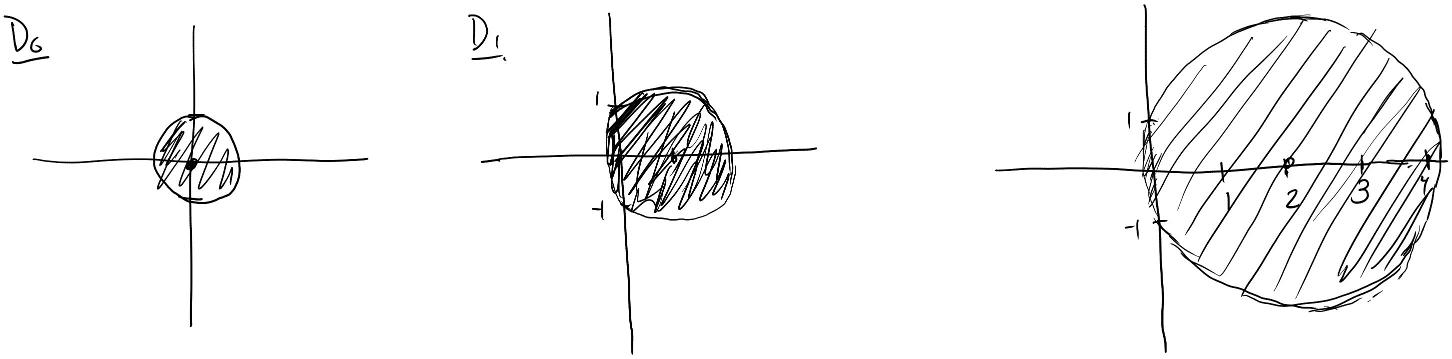
Since the shaded regions are equal,

$(A \cap B) - C = (A - C) \cap B$

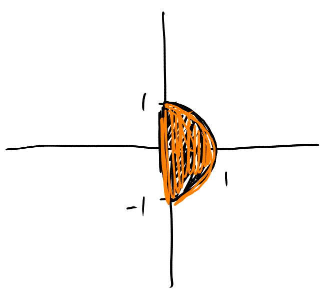
$\boxed{\text{Is valid}}$ for all sets A, B, C.

8. **[3 parts, 6 points each]** Let $D_\alpha = \{(x, y) \in \mathbb{R}^2 \colon (x - \alpha)^2 + y^2 \le \alpha^2 + 1^2\}$. In English, D_α is the closed disk with center at $(\alpha, 0)$ whose circumference passes through the points $(0, -1)$ and $(0, 1)$. Let $I = \{\alpha \in \mathbb{R} \colon \alpha \ge 0\}$.

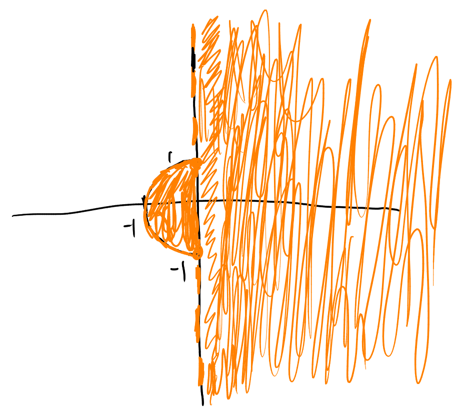
   (a) Sketch the example sets D_0, D_1, and D_2.



   (b) Sketch $\bigcap_{\alpha \in I} D_\alpha$.



   (c) Sketch $\bigcup_{\alpha \in I} D_\alpha$.



9. **[6 points]** Briefly describe Russell's paradox and how mathematicians have addressed it.

Russell's paradox results from defining $R = \{ A : A \notin A \}$ and then observing that both $R \in R$ and $R \notin R$ lead to contradictions. Mathematicians introduced axiomatic set theory which carefully restricts what objects can be collected together to form sets; in particular, it is no longer permitted to define the set R as we have done above.