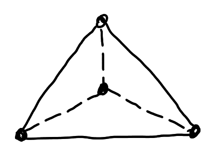
Name: Solutions

Directions: Show all work.

1. Graph Ramsey Numbers.

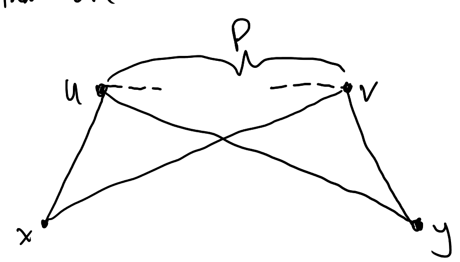
   (a) **[3 points]** Prove that $r(P_4, C_4) > 4$.

   We give a blue (dashed)/red (solid) edge-coloring of K_4 with no blue P_4 and no red C_4:



   (b) **[4 points]** Prove that $r(P_4, C_4) \le 5$.

   We show $K_5 \to P_4, C_4$. Let G be a blue/red edge-coloring of K_5. If G has no blue edges, then we easily find a red C_4. Let P be a blue path in G of maximum length. If P has at least 4 vertices, then we have a blue C_4. Otherwise $2 \le |V(P)| \le 3$. Let u and v be the endpoints of P, and let x and y be 2 vertices in G that are not on P:



   Note that xu and yu must be red, or else P extends to a longer blue path. Similarly, vx and vy must also be red. But now $uyvx$ forms a red 4-cycle. ∎

2. **[3 points]** Using that $r(3,4) = 9$ and $r(3,3,3) = 17$, apply the multicolor Ramsey Theorem to give an upper bound on $r(3,3,4)$.

   From class, $r(n_1, \ldots, n_k) \le 2 + \sum_{i=1}^{k} \left(r(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k) - 1 \right)$. Therefore

$$r(3,3,4) \le 2 + (r(2,3,4) - 1) + (r(3,2,4) - 1) + (r(3,3,3) - 1)$$
$$= -1 + 2r(2,3,4) + r(3,3,3) = -1 + 2r(3,4) + r(3,3,3)$$
$$= -1 + 2 \cdot 9 + 17 = -1 + 18 + 17 = 17 + 17 = \boxed{34}.$$