Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[9 parts, 2 points each]** Decide whether the following are true or false. Indicate your answer by writing the entire word. No justification required.

$A = \{1,2\}$ $B = \{1,2,3\}$ $C = \{1,2,3,\{2,3\}\}$ $D = \{\varnothing,\{1,2,3\}\}$ $E = \{\{2,1\},\{2,3\}\}$

(a) $3 \in B$

$\boxed{\text{TRUE}}$

(b) $A \in E$

$\boxed{\text{TRUE}}$ $\{1,2\}=\{2,1\}$

(c) $\{A\} \in E$

$\boxed{\text{FALSE}}$ $\{\{1,2\}\} \notin E$

(d) $B \in C$

$\boxed{\text{FALSE}}$ Only set in C is $\{2,3\}$

(e) $2 \subseteq A$

$\boxed{\text{FALSE}}$ 2 is not a set

(f) $B \subseteq C$

$\boxed{\text{TRUE}}$

(g) $B \subseteq D$

$\boxed{\text{FALSE}}$ $B \in D$ but $B \not\subseteq D$.

(h) $A \subseteq E$

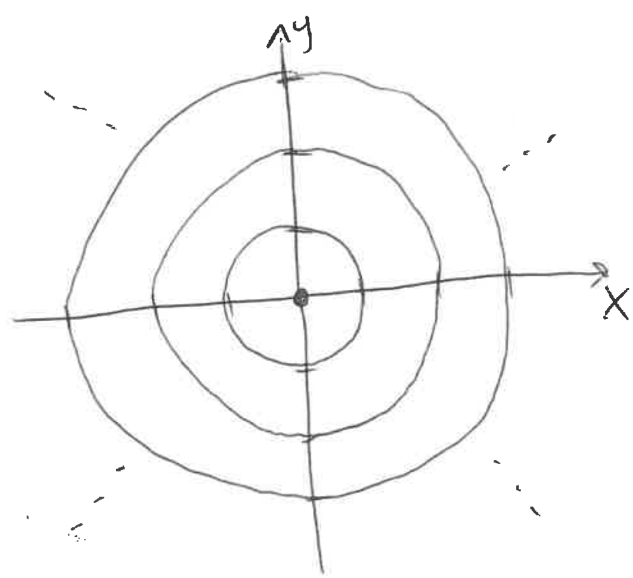
$\boxed{\text{FALSE}}$ $1 \in A$ but $1 \notin E$.

(i) $\{A\} \subseteq E$

$\boxed{\text{TRUE}}$ $A \in E$, so $\{A\} \subseteq E$.

2. **[6 points]** Sketch the set $\{(x,y) \in \mathbb{R}^2\colon \sqrt{x^2+y^2} \in \mathbb{Z}\}$ in the plane. Use dashed lines to indicate boundaries that are omitted from the set.

Note: $\sqrt{x^2+y^2}$ is the distance between (x,y) and the origin $(0,0)$. So the set contains all points at integer distance to the origin. We get concentric circles, centered at the origin, ~~at integer~~ with integer radii. Also, $(0,0)$ is in the set.



3. **[6 parts, 2 points each]** Express each set by listing the elements between braces.

$$A = \{1, \{1,2\}, \{2\}\} \qquad B = \{\varnothing, 2, \{1,2\}, (1,2)\} \qquad C = \{\varnothing, \{2,1\}, (2,1)\}$$

(a) $A \cap B$

$\boxed{\{\{1,2\}\}}$

(b) $B \cap C$

$\boxed{\{\emptyset, \{1,2\}\}}$

Note: $\{1,2\} = \{2,1\}$ but $(1,2) \neq (2,1)$

(c) $(B \cup C) - A$

$\boxed{\{\emptyset, 2, (1,2), (2,1)\}}$

(d) $(B - C) \times (C - B)$

$B - C = \{2, (1,2)\}$

$C - B = \{(2,1)\}$

So $(B-C)\times(C-B) = \boxed{\{(2, (2,1)), ((1,2),(2,1))\}}$

(e) $\mathcal{P}(C - A)$

$C - A = \{\emptyset, (2,1)\}$

$\mathcal{P}(C-A) = \boxed{\{\emptyset, \{\emptyset\}, \{(2,1)\}, \{\emptyset, (2,1)\}\}}$

(f) $(A \cup B \cup C) \cap \mathcal{P}(\mathbb{Z})$

~~These are~~ The members of this set are ~~the elements in A∩B, at least~~ the subsets of the integers that belong to at least one set in $\{A, B, C\}$ So: $\boxed{\{\{1,2\}, \{2\}, \emptyset\}}$.

4. **[6 points]** Is it true or false that $(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$ for all sets A_1, A_2, B_1, and B_2? If true, explain why. If false, give an example where the equality fails.

This is $\boxed{\text{FALSE}}$. For example, let $A_1 = \emptyset$, $A_2 = \{1\}$, $B_1 = \{1\}$, $B_2 = \emptyset$.

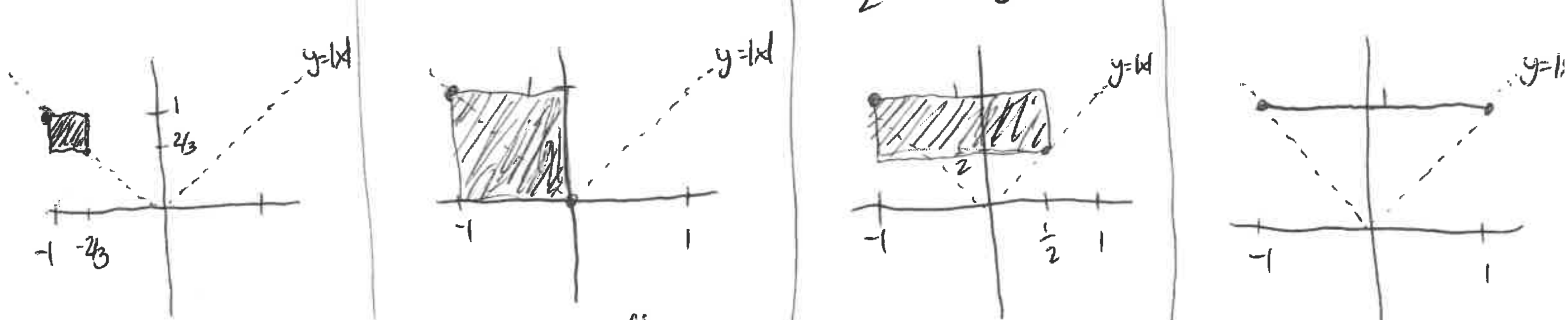
Now $(A_1 \cup A_2) \times (B_1 \cup B_2) = \{1\} \times \{1\} = \{(1,1)\}$ but

$(A_1 \times B_1) \cup (A_2 \times B_2) = \emptyset \cup \emptyset = \emptyset$.

5. **[3 parts, 6 points each]** Recall that when $\alpha \in \mathbb{R}$, we use $|\alpha|$ to denote the absolute value of α. Let $I = [-1,1]$ and for each $\alpha \in I$, let $A_\alpha = [-1,\alpha] \times [|\alpha|, 1]$. In sketches, use dashes to represent omitted boundaries.

   (a) Sketch the example sets $A_{-2/3}$, A_0, $A_{1/2}$, and A_1.

$A_{-2/3} = [-1,-\tfrac{2}{3}]\times[\tfrac{2}{3},1]$ $A_0 = [-1,0]\times[0,1]$ $A_{\frac{1}{2}} = [-1,\tfrac{1}{2}]\times[\tfrac{1}{2},1]$ $A_1 = [-1,1]\times[1,1]$

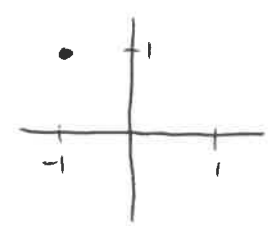


Note: ~~Lower~~ Upper ~~[illegible]~~ left corner always at $(-1,0)$. Lower right corner is along the curve $y=|x|$.

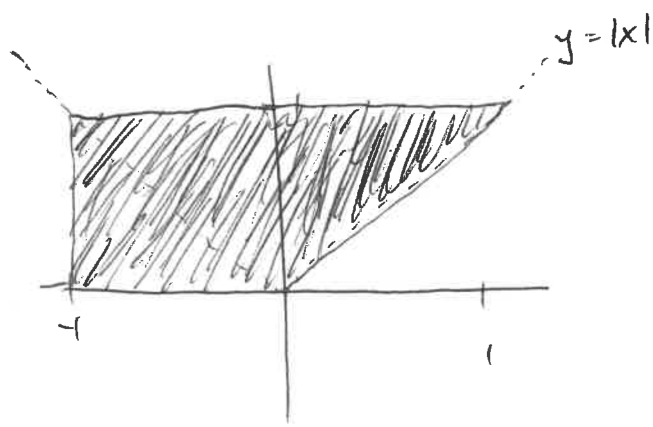
   (b) Sketch $\bigcap_{\alpha\in I} A_\alpha$.

Note: $A_{-1} = [-1,-1]\times[1,1] = \{-1\}\times\{1\} = \{(-1,1)\}$ and $(-1,1)$ belongs to each A_α for $\alpha \in [-1,1]$. So

$$\bigcap_{\alpha\in I} A_\alpha = \{(-1,1)\}.$$



   (c) Sketch $\bigcup_{\alpha\in I} A_\alpha$.



As α goes from -1 to 0, the sets grow until we reach A_0. Then we get the subset of $[0,1]\times[0,1]$ above the line $y=x$ as α goes from 0 to 1.

6. [**3 parts, 4 points each**] Given the open sentences listed below, translate the following English statements into mathematical logic. Then, indicate whether the statement is true or false by writing the entire word.

$P(x)$: x is prime $\quad Q(x)$: x is an even integer $\quad R(x)$: x is a cube integer

(a) The integer 7 is prime but not even.

$P(7) \wedge \sim Q(7)$ TRUE

(b) For 64 to be a cube number, it is necessary that 64 is not prime.

$R(64) \Rightarrow \sim P(64)$ TRUE

(c) For an integer 7 to be even, it is sufficient for 11 to be prime.

$P(11) \Rightarrow Q(7)$ FALSE

7. [**2 parts, 6 points each**] Let φ be the statement $((P \wedge Q) \vee \sim P) \Rightarrow (P \Leftrightarrow Q)$.

(a) Give a truth table for φ.

P	Q	$P \wedge Q$	$\sim P$	$(P \wedge Q) \vee \sim P$	$P \Leftrightarrow Q$	φ
T	T	T	F	T	T	T
T	F	F	F	F	F	T
F	T	F	T	T	F	F
F	F	F	T	T	T	T

(b) Find a simple statement which is logically equivalent to φ.

Note: φ ~~is true except~~ fails only when Q is true and P is false.

So $\varphi \equiv \boxed{Q \Rightarrow P}$.

8. **[2 parts, 4 points each]** Translate the following statements from formal mathematical logic to English, as naturally and efficiently as possible. Then, indicate whether the statement is true or false by writing the entire word.

(a) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 0.$

There is a real number x such that no matter what real number is added to x, the result is 0. This is $\boxed{\text{FALSE}}$: we can add only one number to x and get zero.

(b) $\forall X \subseteq \mathbb{N}, [X \neq \varnothing \Rightarrow (\exists m \in X, \forall y \in X, y \geq m)].$

Every nonempty subset X of the natural numbers contains a minimum element m. This is $\boxed{\text{TRUE}}$.

9. **[2 parts, 4 points each]** Negate the following sentences, as naturally and efficiently as possible.

(a) For some real number x, we have $x^2 = 2$.

Soln 1: For each real number x, we have $x^2 \neq 2$.

Soln 2: The value $\sqrt{2}$ is not real.

(b) For each real number x, at least one of $\{\sin(x), \cos(x), \tan(x)\}$ is positive.

There is a real number x such that all of $\{\sin(x), \cos(x), \tan(x)\}$ are at most zero.