Math 283 Quiz 8 Fri. Mar. 9, 2018

Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2 points]** Find the coefficient of x^5 in $(2-x)^8$. (It is small enough that you will be able to do the computation by hand.)

Note that $(2-x)^8 = \sum_{k=0}^{8} \binom{8}{k} 2^k (-x)^{8-k}$.

All contributions to x^5 come from the term $k=3$.

We compute $\binom{8}{3} \cdot 2^3 (-x)^{8-3}$:

$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} \cdot 8 \cdot (-1) \cdot x^5$

$= -8 \cdot 8 \cdot 7 \cdot x^5$

$64 \cdot 7 = 448$

$= -448\, x^5$.

The coefficient is $\boxed{-448}$.

2. **[2 points]** Use the Binomial Theorem to find a simple formula for $\sum_{k=0}^{n} \binom{n}{k} 8^k (-5)^{n-k}$. Your answer may involve factorials and/or binomial coefficients.

We have $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$. With $x=8$ and $y=-5$,

this gives $\sum_{k=0}^{n} \binom{n}{k} 8^k (-5)^{n-k} = (8+(-5))^n = \boxed{3^n}$.

DROP

3. **[1 point]** Find the coefficient of x^2y^2 in $(x+1)^4(y+1)^6$.

Applying the Binomial Theorem twice, we have ~~Note that~~

$$(x+1)^4(y+1)^6 = \left(\sum_{k=0}^{4} \binom{4}{k} x^k\right)\left(\sum_{l=0}^{6} \binom{6}{l} y^l\right).$$

All contributions to x^2y^2 arise when $k=l=2$; these terms generate the product $\binom{4}{2}x^2 \cdot \binom{6}{2} \cdot y^2$. Therefore the coefficient is $\binom{4}{2} \cdot \binom{6}{2}$, and $\binom{4}{2}\binom{6}{2} = \frac{4 \cdot 3}{2 \cdot 1} \cdot \frac{6 \cdot 5}{2 \cdot 1} = 6 \cdot 15 = \boxed{90}$.

4. [~~2~~ 3 **points**] Let $n \in \mathbb{N}$. Prove that if n is odd and $\binom{n}{2}$ is even, then $n \equiv 1 \pmod 4$.

Suppose that n is odd and $\binom{n}{2}$ is even. Since n is odd, we have that $n = 2t+1$ for some $t \in \mathbb{Z}$. Observe that

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{(2t+1)((2t+1)-1)}{2} = \frac{(2t+1)(2t)}{2} = t(2t+1).$$

~~Since $\binom{n}{2}$ is even, we have $t(2t+1) = 2l$ for some $l \in \mathbb{Z}$.~~

It follows that $t(2t+1)$ is even. Since the product of two odd integers is odd and $2t+1$ is odd, it must be t is even (or else $t(2t+1)$ would be odd.) Since t is even, it follows that $t = 2k$ for some $k \in \mathbb{Z}$. We have $n = 2t+1 = 2(2k)+1 = 4k+1$, and so $n-1 = 4k$. It follows that $4 \mid n-1$, and this means that $n \equiv 1 \pmod 4$. ☒

5. [**3 parts, 1 point each**] Let A be a set of size $2n$. Answer each question with a simple formula in terms of n; your formula may involve factorials and/or binomial coefficients. No justification required.

(a) How many subsets of A are there?

$$2^{|A|} = \boxed{2^{2n}} = \boxed{4^n}$$

(b) How many subsets of A contain exactly half the elements in A?

These are the n-element subsets of a set of size $2n$, so there are $\boxed{\binom{2n}{n}}$ of these.

(c) How many subsets of A contain fewer than half the elements in A?

If we throw away all the ~~subsets~~ n-element subsets, then half of the remaining subsets have size less than n and half have size more than n. So there are $\boxed{\frac{1}{2}\left[2^{2n} - \binom{2n}{n}\right]}$ of these.