Math 283 Quiz 10 Fri. Apr. 6, 2018

Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[3 points]** Suppose $a, b \in \mathbb{Z}$. Prove that if ab is odd, then $a^2 + b^2$ is even.

Suppose that ab is odd. Since an even integer times any integer is even, it follows that both a and b are odd. Therefore $a = 2k+1$ and $b = 2l+1$ for some $k, l \in \mathbb{Z}$. We compute

$$a^2+b^2 = (2k+1)^2 + (2l+1)^2 = 4k^2+4k+1+4l^2+4l+1$$
$$= 2(2k^2+2k+2l^2+2l+1).$$

Since $2 \mid a^2+b^2$, we have that a^2+b^2 is even.

2. **[3 points]** Prove that there exist unique real numbers a and b such that the linear function f given by $f(x) = ax + b$ satisfies $f(f(x)) = x + 1$.

Existence: Let $a=1$ and $b=\frac{1}{2}$. We have $f(f(x)) = f(x+\frac{1}{2}) = (x+\frac{1}{2})+\frac{1}{2}$ and so $f(f(x)) = x+1$. It follows that such real numbers exist.

Uniqueness: Suppose that $f(x) = ax+b$ and $f(f(x)) = x+1$ for some $a, b \in \mathbb{R}$. We have $f(ax+b) = x+1$, and so $a(ax+b)+b = x+1$. Therefore $a^2x + ab + b = x+1$. Since the slopes of these linear functions must be equal, we have that $a^2 = 1$, and so $a=-1$ or $a=1$. If $a=-1$, then $a^2x+ab+b = x+1$ becomes $x - b + b = x+1$, or $x = x+1$, which is impossible. Therefore $a=1$ and we have $x+2b = x+1$. It follows that $b=\frac{1}{2}$.

3. **[4 points]** Let A, B, and C be sets. Show that $A \times B \subseteq A \times C$ if and only if $A = \varnothing$ or $B \subseteq C$.

Suppose that $A \times B \subseteq A \times C$. If $A = \emptyset$, then we are done. So suppose A is non-empty and let $a \in A$. We show $B \subseteq C$. Suppose $b \in B$. Since $a \in A$ and $b \in B$, we have $(a, b) \in A \times B$. Since $A \times B \subseteq A \times C$, we have that $(a, b) \in A \times C$. Therefore $b \in C$. Since every element in B is also in C, we have $B \subseteq C$.

Conversely, suppose that $A = \emptyset$ or $B \subseteq C$. If $A = \emptyset$, then $A \times B = \emptyset$ and $A \times C = \emptyset$. Of course, $\emptyset \subseteq \emptyset$ and so $A \times B \subseteq A \times C$ in this case. Otherwise, suppose if $A \neq \emptyset$, we have that $B \subseteq C$. ~~and~~ Suppose $(a, b) \in A \times B$. This means that $a \in A$ and $b \in B$. Also, since $B \subseteq C$, we have that $b \in C$. Since $a \in A$ and $b \in C$, we have that $(a, b) \in A \times C$. Since every element in $A \times B$ is also in $A \times C$, we conclude that $A \times B \subseteq A \times C$.