Directions: Solve 5 of the following 6 problems. All written work must be your own, and you must cite any external sources that you use.

1. [MR 2.24] Prove that with high probability, $G_{2k,1/2}$ contains a perfect matching. (Hint: partition the vertices into sets A and B of size k. Using only randomness in pairs with one vertex in A and the other in B, obtain a matching with $2k/3$ edges. Next, using randomness inside A and inside B, modify the existing matching to obtain a perfect matching.)

2. *Expected time and distance.*

   (a) Starting in the center of a path on $2r-1$ vertices, a random walk moves in either direction with probability $1/2$ independently of all previous steps. The walk ends when moving off of either end of the path. Compute the expected number of steps in the walk. (Hint: recall that for a path on k vertices, a random walk starting at an endpoint lasts for k steps on average.)

   (b) Let $X_1, \ldots, X_{2k+1}$ be independent random variables with each X_i chosen uniformly in $\{-1, +1\}$. Let $X = \sum_i X_i$. Obtain $\mathbb{E}(|X|)$ in closed form (i.e. not as a sum), and use Stirling's formula to obtain the asymptotic behavior. Note: this is the expected distance between the starting and ending positions of a random walk on a long path with $2k+1$ steps. (Hint: a recurrence leads to a non-trivial (but solvable) sum. For a faster solution, let Y_i be the random variable which is 1 if X_i and X have the same sign and -1 otherwise.)

3. Let S be chosen uniformly at random from $\binom{[n]}{k}$ and let $X = \min S$. Compute $\mathbb{E}(X)$.

4. Large chromatic number and clique size.

   (a) [MR 3.6] Show that for n sufficiently large, there exists an n-vertex graph with chromatic number at least $n/2$ and $\omega(G) \le O(n^{2/3} \log n)$.

   (b) Let $\varepsilon > 0$. Prove that if G is an n-vertex graph and $\chi(G) \ge (\frac{1}{2} + \varepsilon)n$, then $\omega(G) \ge 2\varepsilon n$.

5. *Sperner's theorem.* Let $\mathcal{F}$ be a family of subsets of $[n]$, and suppose that $\mathcal{F}$ does not contain distinct sets A and B such that $A \subseteq B$.

   (a) Prove that $\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \le 1$. Hint: consider a random ordering of $[n]$.

   (b) Conclude from part (a) that $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$.

   Comment: since $\binom{[n]}{\lfloor n/2 \rfloor}$ satisfies the given condition, the inequality in (b) is sharp.

6. [AS 1.7] Let $\{(A_1, B_1), \ldots, (A_n, B_n)\}$ be a collection of ordered pairs of sets with $|A_i| = r$ and $|B_i| = s$ for all i. Suppose that A_i and B_i are disjoint for all i, but whenever $i \ne j$, at least one of $\{A_i \cap B_j, A_j \cap B_i\}$ is nonempty. Prove that $n \le \frac{(r+s)^{r+s}}{r^r s^s}$.