Directions: Solve 5 of the following 6 problems. Students seeking higher than 500 level credit must complete all 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines that apply to all homeworks.

1. [CM 1.2.{16,15}] Prove the first three identities below by counting a set in two ways. In each case, give a single direct argument without manipulating the formulas. In part (d), find a closed form solution for the sum and give a combinatorial proof.

   (a) $\binom{2n}{n} = 2\binom{2n-1}{n-1}$

   (b) $\sum_k \binom{k}{l}\binom{n}{k} = \binom{n}{l}2^{n-l}$

   (c) $\sum_{i=1}^{n} i(n-i) = \sum_{i=1}^{n} \binom{i}{2}$

   (d) $\sum_{j=1}^{m}(m-j)2^{j-1}$

2. [CM 1.2.19] Evaluate the following sums using known identities.

   (a) $\sum_{k\geq 0} \frac{1}{k+1}\binom{n}{k}$

   (b) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{n+1-k}$

3. [CM 1.2.20] Give a combinatorial proof for the following identity by devising a set that both sides count.

$$\sum_{k\geq 1} k\binom{m+1}{r+k+1} = \sum_{i=1}^{m} i2^{i-1}\binom{m-i}{r}$$

4. [CM 1.2.28] Evaluate the sums below. (Hint: Express each sum as a product.)

   (a) $\sum_{S\subseteq[n]} \prod_{i\in S} \frac{1}{i}$

   (b) $\sum_{S\subseteq[n]} (-1)^{|S|} \prod_{i\in S} \frac{1}{i}$

5. [CM 1.2.30] For the identity below,

   (a) Give a combinatorial proof by constructing a set that both sides count.

   (b) Use the Binomial Theorem to prove that both sides arise as the coefficient of x^n in the expansion of $(1+3x+x^2)^n$. Hint: if you are having trouble generating the LHS with the Binomial Theorem, try to apply your combinatorial argument from (a) in the algebraic context. What plays the role of j in the combinatorial argument? What is the corresponding mechanism in the algebraic context?

$$\sum_{j\geq 0}\binom{n}{j}\binom{2j}{j} = \sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k}3^{n-2k}$$

6. [CM 1.2.39] Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}$, with $a_1 < \cdots < a_r$. For $i \in [n]$, let S_i be the family of i-element subsets of A, and let $\sigma_i = \sum_{B\in S_i} \max(B)$. For $k \in [n]$, prove that

$$a_k = \sum_{r=1}^{n}(-1)^{k+r}\binom{r-1}{k-1}\sigma_r.$$