Name: Solutions

Directions: Show all work. No credit for answers without work. This test has 50 available points, but will be graded out of 40 points (scores capped at 40). In addition, there is one 4 point bonus question (bonus points are not capped).

1. [**8 points**] Find the general solution to $4y'' + 4y' + y = 3xe^x$.

$$4r^2 + 4r + 1 = 0$$
$$(2r+1)(2r+1) = 0$$
$$r = -\tfrac{1}{2}$$
$$\boxed{Y_c = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}}$$

Terms: xe^x

$e^x + \cancel{xe^x}$

• Since these are independent from Y_c:

$$Y_p = Axe^x + Be^x = e^x(Ax+B)$$

• $Y_p' = e^x(Ax+B) + e^x \cdot A$
$= e^x(Ax + (A+B))$

• $Y_p'' = e^x(Ax + (A+B)) + e^x \cdot A$
$= e^x(Ax + (2A+B))$

• e^x: $4(2A+B) + 4(A+B) + B = 0$
$12A + 9B = 0$; $4 + 9B = 0$, $B = -\frac{4}{9}$

• xe^x: $4A + 4A + A = 3$
$9A = 3$, $A = \frac{1}{3}$

$$\boxed{Y = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x} + \tfrac{1}{3}xe^x - \tfrac{4}{9}e^x}$$

2. [**5 points**] Find the form of a particular solution y_p to $y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$. Note: do not actually solve for the constants in y_p.

$$r^5 - r^3 = 0$$
$$r^3(r^2 - 1) = 0$$
$$r^3(r+1)(r-1) = 0$$
$(r = 0)$ 3, $r = -1$, $r = 1$

~~$Y_c = A + x + x^2 + e$~~

Y_c: $1, x, x^2, e^x, e^{-x}$

RHS	e^x	$2x^2$	1
	↓ e^x	↓ $2x$ ↓ 2	
mult by:	x	x^3	x^3

$$\boxed{Y_p = Axe^x + Bx^3 + Cx^4 + Dx^5}$$

3. Laplace Transform.

   (a) [**1 point**] State the definition of the Laplace Transform of a function $f(t)$.

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

   (b) [**4 points**] Use the definition to prove that $\mathcal{L}\{1\} = \frac{1}{s}$ for $s > 0$.

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} \cdot 1\,dt = \left(-\frac{1}{s}\right)e^{-st}\Big|_0^{\infty}$$

$$= \lim_{b\to\infty}\left(\left(-\frac{1}{s}e^{-sb}\right) - \left(-\frac{1}{s}\cdot e^{0}\right)\right)$$

$$= \lim_{b\to\infty}\left(\frac{1}{s} - \cancel{\frac{1}{se^{sb}}}^{\to 0}\right) = \boxed{\frac{1}{s}}$$

4. [**4 parts, 2 points each**] Find the Inverse Laplace Transform of the following.

   (a) $F(s) = \frac{s+1}{s^2+9}$

$$= \frac{s}{s^2+9} + \frac{1}{s^2+9}$$

$$= \frac{s}{s^2+9} + \frac{1}{3}\cdot\frac{3}{s^2+9}$$

$$\boxed{f(t) = \cos(3t) + \frac{1}{3}\sin(3t)}$$

   (b) $F(s) = \frac{1}{(s+2)(s+3)}$

$$= \frac{A}{s+2} + \frac{B}{s+3}$$

$$A(s+3) + B(s+2) = 1$$

$s=-3$: $B(-1) = 1$, $B = -1$

$s=-2$: $A(1) = 1$

$$F(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

$$\boxed{f(t) = e^{-2t} - e^{-3t}}$$

   (c) $F(s) = \frac{2s+6}{(s+1)^5}$

$$= \frac{2(s+1)+4}{(s+1)^5}$$

$$f(t) = e^{-t}\mathcal{L}^{-1}\left\{\frac{2s+4}{s^5}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{2}{s^4} + \frac{4}{s^5}\right\}$$

$$= e^{-t}\left(\frac{2}{3!}t^3 + \frac{4}{4!}t^4\right)$$

$$= \boxed{t^3e^{-t}\left(\frac{1}{3} + \frac{1}{6}t\right)}$$

   (d) $F(s) = \frac{1}{s^2+8s+25}$

$$= \frac{1}{(s+4)^2+9}$$

$$f(t) = e^{-4t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= e^{-4t}\mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{3}{s^2+9}\right\}$$

$$= \boxed{e^{-4t}\cdot\frac{1}{3}\sin(3t)}$$

5. [**12 points**] Use the Laplace Transform to solve the following IVP.

$$x'' + 6x' + 25x = 0 \qquad \text{with } x(0) = 2 \text{ and } x'(0) = 3$$

$$\left[s^2X - s\,x(0) - x'(0)\right] + 6\left[sX - x(0)\right] + 25X = 0$$

$$s^2X - 2s - 3 + 6sX - 12 + 25X = 0$$

$$(s^2 + 6s + 25)X = 2s + 15$$

$$X = \frac{2s+15}{s^2+6s+25} = \frac{2s+15}{(s+3)^2+16} = \frac{2(s+3)+9}{(s+3)^2+16}$$

$$x(t) = e^{-3t}\mathcal{L}^{-1}\left\{\frac{2s+9}{s^2+16}\right\}$$

$$= e^{-3t}\mathcal{L}^{-1}\left\{2\cdot\frac{s}{s^2+16} + \frac{9}{4}\cdot\frac{4}{s^2+16}\right\}$$

$$= \boxed{e^{-3t}\left(2\cos(4t) + \frac{9}{4}\sin(4t)\right)}$$

6. [**12 points**] Use the Laplace Transform to solve the following system of equations.

$$x' = x + 2y \qquad x(0) = y(0) = 0$$
$$y' = x + e^{t}$$

$$sX = X + 2Y \qquad\qquad (s-1)X - 2Y = 0$$
$$sY = X + \frac{1}{s-1} \qquad\qquad -X + sY = \frac{1}{s-1}$$

$$\begin{vmatrix} s-1 & -2 \\ -1 & s \end{vmatrix} = s(s-1) - (-2)(-1) = s^2 - s - 2 = (s-2)(s+1)$$

$$X = \frac{\begin{vmatrix} 0 & -2 \\ \frac{1}{s-1} & s \end{vmatrix}}{(s-2)(s+1)} = \frac{2}{(s-2)(s+1)(s-1)} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s-1}$$

$$Y = \frac{\begin{vmatrix} s-1 & 0 \\ -1 & \frac{1}{s-1} \end{vmatrix}}{(s-2)(s+1)} = \frac{1}{(s-2)(s+1)} = \frac{D}{s-2} + \frac{E}{s+1}$$

$\underline{X}$: $A(s+1)(s-1) + B(s-2)(s-1) + C(s-2)(s+1) = 2$

$\underline{s=1}$: $C(-1)(2) = 2$, $C = -1$

$\underline{s=-1}$: $B(-3)(-2) = 2$, $B = \frac{1}{3}$

$\underline{s=2}$: $A(3)(1) = 2$, $A = \frac{2}{3}$

$$X(s) = \frac{2}{3} \cdot \frac{1}{s-2} + \frac{1}{3}\frac{1}{s+1} - \frac{1}{s-1}$$

$$\boxed{x(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} - e^{t}}$$

$\underline{Y}$: $D(s+1) + E(s-2) = 1$

$\underline{s=-1}$: $E(-3) = 1$, $E = -\frac{1}{3}$

$\underline{s=2}$: $D(3) = 1$, $D = \frac{1}{3}$

$$Y(s) = \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1}$$

$$\boxed{y(t) = \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}}$$

Scratch Space

7. [**4 bonus points**] Find the Inverse Laplace Transform of $F(s) = \frac{1}{(s^2+1)(s+1)}$.

$$F(s) = \frac{As+B}{s^2+1} + \frac{C}{s+1}$$

$$As+B + \frac{C(s^2+1)}{s+1} = \frac{1}{s+1}$$

$\underline{s=i}$: $Ai + B + 0 = \frac{1}{i+1}$

$$Ai + B = \frac{1}{i+1} \cdot \frac{i-1}{i-1}$$

$$Ai + B = \frac{i-1}{i^2-1} = \frac{i-1}{-2} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$$

$\underline{\text{Real} = \text{Real}}$: $B = \frac{1}{2}$

$\underline{\text{Imag} = \text{Imag}}$ $A = -\frac{1}{2}$

$$(s+1)\left(\frac{As+B}{s^2+1}\right) + C = \frac{1}{s^2+1}$$

$\underline{s=-1}$: $0 + C = \frac{1}{2}$

$$F(s) = -\frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s+1}$$

$$\boxed{f(t) = -\frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) + \frac{1}{2}e^{-t}}$$