Name: Key

1. [~~2~~ 3 **points**] Consider the following argument. Hypotheses: If the refrigerator is cold, then it is plugged in. The refrigerator is unplugged if and only if it is quiet. The refrigerator is noisy. Conclusion: The refrigerator is cold.

   Translate this argument into a single wff. Use $C$ for "the refrigerator is cold", $P$ for "the refrigerator is plugged in", and $Q$ for "the refrigerator is ~~noisy~~ quiet".

$$(C \to P) \wedge (P' \leftrightarrow Q) \wedge Q' \to C$$

2. Two parts.

   (a) [~~5~~ 6 **points**] Write a truth table for the following wff:

$$P:\quad ((A \wedge B) \leftrightarrow C) \overset{\vee}{\cancel{\wedge}} ((C \to B') \vee A)$$

| A | B | C | $A \wedge B$ | $(A \wedge B) \leftrightarrow C$ | $C \to B'$ | $(C \to B') \vee A$ | P |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | T | T |
| T | T | F | T | F | T | T | T |
| T | F | T | F | F | T | T | T |
| T | F | F | F | T | ~~F~~ T | T | T |
| F | T | T | F | F | F | F | F |
| F | T | F | F | T | T | T | T |
| F | F | T | F | F | T | T | T |
| F | F | F | F | T | T | T | T |

   (b) [**1 point**] Is the wff a tautology? Briefly explain why or why not.

No, since the sentence is false when A is false and B and C are true.

| Derivation Rule | Name/Abbreviation for Rule |
|---|---|
| $P \vee Q \iff Q \vee P$<br>$P \wedge Q \iff Q \wedge P$ | Commutative—comm |
| $(P \vee Q) \vee R \iff P \vee (Q \vee R)$<br>$(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$ | Associative—ass |
| $(P \vee Q)' \iff P' \wedge Q'$<br>$(P \wedge Q)' \iff P' \vee Q'$ | De Morgan's laws—De Morgan |
| $P \to Q \iff P' \vee Q$ | Implication—imp |
| $P \iff (P')'$ | Double negation—dn |
| $P \leftrightarrow Q \iff (P \to Q) \wedge (Q \to P)$ | Defn of Equivalence—equ |
| $\left.\begin{array}{c} P \\ P \to Q \end{array}\right\} \implies Q$ | Modus ponens—mp |
| $\left.\begin{array}{c} P \to Q \\ Q' \end{array}\right\} \implies P'$ | Modus tollens—mt |
| $\left.\begin{array}{c} P \\ Q \end{array}\right\} \implies P \wedge Q$ | Conjunction—con |
| $P \wedge Q \implies \left\{\begin{array}{c} P \\ Q \end{array}\right.$ | Simplification—sim |
| $P \implies P \vee Q$ | Addition—add |

3. **[6 points]** Using only the given derivation rules, give a proof sequence to show the following wff is a tautology.

$$(P \vee Q) \wedge P' \to Q$$

| | | |
|---|---|---|
| 1. | $P \vee Q$ | hyp |
| 2. | $(P')' \vee Q$ | 1, dn |
| 3. | $P' \to Q$ | 2, mp |
| 4. | $P'$ | hyp |
| 5. | $Q$ | 3,4 mp |

Spearing.

4. [~~6~~ 5 **parts, 2 points each**] Determine whether the following sentences are valid. If the sentence is not valid, give an interpretation under which the sentence is false. Clearly indicate the domain of any interpretation you give.

(a) $\exists x\,[Q(x)]$

Not valid.
Domain = integers
$Q(x)$: $x = \pi$

Any Q which is false for all objects in the domain is OK.

(b) $(\exists x\,[Q(x)] \to \forall x\,[Q(x)]) \to \exists x\,[(Q(x))']$

Not valid.
Domain = integers.
$Q(x)$: $x^2 \geq 0$.

Any Q which is true for all objects in the domain is OK.

(c) $\forall x\,[\exists y\,[P(x,y)]] \to \exists x\,[\forall y\,[P(x,y)]]$

Not Valid.
Domain = integers
$P(x,y)$: $x + y = 0$

(d) $\exists x\,[\forall y\,[P(x,y)]] \to \forall x\,[\exists y\,[P(x,y)]]$

Not Valid.
Domain = integers
$P(x,y)$: $x = 0$

(e) $\exists x\,[\forall y\,[P(x,y)]] \to \forall y\,[\exists x\,[P(x,y)]]$

Valid.

~~(f) $\forall x\,[\exists y\,[P(x,y)]] \to \exists y\,[\forall x\,[P(x,y)]]$~~

5. [~~6~~ 5 **points**] Prove that the sum of three odd integers is odd.

Proof. Let $x$, $y$, and $z$ be odd integers. Because $x$, $y$, and $z$ are odd, $x = 2r+1$, $y = 2s+1$, and $z = 2t+1$ for some integers $r$, $s$, and $t$.

Therefore
$$\begin{aligned} x+y+z &= (2r+1)+(2s+1)+(2t+1) \\ &= 2(r+s+t)+3 \\ &= 2(r+s+t+1)+1 \end{aligned}$$

and since $r+s+t+1$ is an integer, $x+y+z$ is odd.

6. [~~6~~ 5 **points**] Prove that if the product of two integers $x$ and $y$ is odd, then $x$ is odd and $y$ is odd.

Proof: By contraposition. We prove that if $x$ is even or $y$ is even, then $xy$ is even.

Case 1: $x$ is even. Since $x$ is even, $x = 2r$ for some integer $r$. Therefore $xy = (2r)y = 2(ry)$, and so $xy$ is even.

Case 2: $y$ is even. Since $y$ is even, $y = 2s$ for some integer $s$. Therefore $xy = x(2s) = 2(xs)$, and so $xy$ is even.

In both cases, $xy$ is even.

7. [10] Prove by induction that $3^{3n} - 1$ is divisible by 13 for every positive integer $n$.

Proof: By induction.

Basis Step: If $n = 1$, then $3^{3 \cdot n} - 1 = 3^3 - 1 = 26$, which is divisible by 13.

Inductive Step: Let $n \geq 2$. Note that $3^{3n} - 1$ equals $3^3 \cdot 3^{3(n-1)} - 1$. By the I.H., $3^{3(n-1)} - 1 = 13k$ for some integer $k$, and so $3^{3(n-1)} = 13k + 1$. Therefore

$$\begin{aligned} 3^{3n} - 1 &= 3^3 \cdot 3^{3(n-1)} - 1 \\ &= 3^3 (13k + 1) - 1 \\ &= 27 \cdot 13k + 27 - 1 \\ &= 27 \cdot 13k + 26 \\ &= 13 \cdot (27k + 2). \end{aligned}$$

Since $27k + 2$ is an integer, $3^{3n} - 1$ is divisible by 13. ☐

8. [~~**6**~~ 10. **points**] Prove by induction that any amount of postage greater than or equal to 12 cents can be buil~~d~~t using only 3-cent and 5-cent stamps.

Proof: By induction.

Basis Step: Since

$$12 = 4\cdot 3 + 0\cdot 5,$$
$$13 = 1\cdot 3 + 2\cdot 5, \quad \text{and}$$
$$14 = 3\cdot 3 + 1\cdot 5,$$

we can build $n$ cents of postage when $12 \le n \le 14$.

Inductive Step: Let $n \ge 15$. Because $12 \le n-3 < n$, the I.H. implies that we can build $n-3$ cents of postage with 3-cent and 5-cent stamps. Therefore $n-3 = x\cdot 3 + y\cdot 5$ for some non-negative integers $x$ and $y$. Hence $n = (x+1)\cdot 3 + y\cdot 5$, and so $n$ cents of postage can be built with $(x+1)$ 3-cent stamps and $y$ 5-cent stamps.

QED