Name: Key

Unless told otherwise, show your work. Answers without work earn reduced credit.

1. [**3 points**] Decide whether the given functions are one-to-one/injective, onto/surjective, or bijective. For each blank cell in the table, write "Yes" if the function has the property, and "No" otherwise. You do not need to show your work.

   In the following, let A^* be the set of finite strings of a's and b's. For example, $aaba$, bb, and the empty string λ are all in A^*. Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}$ is the set of integers.

Function	one-to-one	onto	bijective
$f\colon \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$	Yes	Yes	Yes
$f\colon \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 - 1$	No	No	No
$f\colon \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^3 - 1$	Yes	No	No
$f\colon A^* \to \mathbb{N}$ where $f(x)$ equals the length of x	No	Yes	No
$f\colon A^* \to A^*$ where $f(x) = xx$	Yes	No	No
$f\colon A^* \to A^*$ where $f(x)$ equals the reverse of x	Yes	Yes	Yes

2. [**2 parts, 1 point each**] Let A be the set of all strings of a's and b's of length 8. Let $f\colon A \to A$ be the function that shifts every character to the right, and moves the 8th character to the front of the string. For example, $f(abaaabbb) = babaaabb$. Let $g\colon A \to A$ be the function that reverses the string. For example, $g(babaaabb) = bbaaabab$.

   (a) Find $(f \circ g)(abbababb)$.

   $$f(g(abbababb)) = f(bbababba)$$
   $$= \boxed{abbababb}$$

   (b) Let $h = f \circ g$. Is h a bijection? If h is a bijection, describe the inverse h^{-1}. If h is not a bijection, explain why.

   Yes, h is a bijection. The inverse h^{-1} first shifts everything character to the left and moves the first character to the last position, and second h^{-1} reverses the string. Also ok: $\boxed{h^{-1} = g^{-1} \circ f^{-1}}$.

3. **[3 parts, 1 point each]** Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Express the following permutations as the composition of zero or more disjoint cycles; each cycle should have at least 2 elements.

(a) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 6 & 1 & 4 & 3 & 8 & 7 \end{pmatrix}$

$$\boxed{(1\ 5\ 4) \circ (3\ 6) \circ (7\ 8)}$$

(b) $(3\ 6\ 8\ 4) \circ (6\ 2\ 4\ 5)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 6 & 5 & 8 & 2 & 7 & 4 \end{pmatrix}$$

$$\boxed{(2\ 3\ 6) \circ (4\ 5\ 8)}$$

(c) $(2\ 6\ 8) \circ (2\ 7) \circ (3\ 1\ 6\ 5) \circ (4\ 2)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 1 & 7 & 3 & 5 & 6 & 2 \end{pmatrix}$$

$$\boxed{(1\ 8\ 2\ 4\ 7\ 6\ 5\ 3)}$$

4. **[2 points]** Prove, by finding constants that satisfy the definition of order of magnitude, that $f = \Theta(g)$ if $f(n) = 3\log(n^5)$ and $g(n) = \log(n)$.

$f = O(g)$: $f(n) = 3 \cdot 5 \log(n) = 15 \log(n)$.

$15 \log(n) \le c \log(n)$ when $\boxed{n \ge 1 \text{ and } c = 15}$

$g = O(f)$: $\log(n) \le c \cdot 15 \cdot \log(n)$ when $\boxed{n \ge 1 \text{ and } c = 1}$