Name: Key

1. Find the minimum value of $f(x,y) = x^2 + 4y^2 - 2xy$ subject to $2x + y = \cancel{10}\ 14$. (Compare with 7.5 #4.)

$\cdot f_x(x,y) = 2x - 2y$

$\cdot f_y(x,y) = 8y - 2x$

$\cdot g_x(x,y) = 2$

$\cdot g_y(x,y) = 1$

$\underline{f_x = \lambda g_x}$: $2x - 2y = \lambda \cdot 2$

$x - y = \lambda$

$\underline{f_y = \lambda g_y}$: $8y - 2x = \lambda \cdot 1$

$8y - 2x = \lambda = x - y$

$9y = 3x$

$3y = x$

$\cdot$ $\underline{g = k}$: $2x + y = 14$

$2(3y) + y = 14$

$7y = 14$

$y = 2$

$\cdot$ $x = 3y = 3 \cdot 2 = 6$

$\cdot f(6,2) = 6^2 + 4 \cdot 2^2 - 2 \cdot 6 \cdot 2$

$= 6^2 + 4(4 - 6)$

$= 36 + 4(-2)$

$= 36 - 8 = \boxed{28}$

So min. value is $\boxed{28}$.

2. Find the maximum and minimum values of $f(x,y) = 4x^2 - 6xy + y^2$ subject to the constraint $4x^2 + y^2 = 1$. (Compare with 7.5 #6.)

$\cdot f_x(x,y) = 8x - 6y$

$\cdot f_y(x,y) = -6x + 2y$

$\cdot g_x(x,y) = 8x$

$\cdot g_y(x,y) = 2y$

$\underline{f_x = \lambda g_x}$: $8x - 6y = \lambda 8x$

$1 - \frac{6y}{8x} = \lambda$

$1 - \frac{3y}{4x} = \lambda$

$\underline{f_y = \lambda g_y}$: $-6x + 2y = \lambda 2y$

$\frac{-6x}{2y} + 1 = \lambda$

$\lambda = 1 - \frac{3x}{y}$

$1 - \frac{3x}{y} = 1 - \frac{3y}{4x}$

$\cancel{1} + \frac{3y}{4x} = \cancel{1} + \frac{3x}{y}$

$3y^2 = 12x^2$

$y^2 = 4x^2$

$\underline{g = k}$: $4x^2 + y^2 = 1$

$4x^2 + 4x^2 = 1$

$8x^2 = 1$

$x^2 = \frac{1}{8}$

$x = \pm \frac{1}{\sqrt{8}}$

$\underline{x = -\frac{1}{\sqrt{8}}}$: $y^2 = 4x^2$

$y^2 = 4\left(-\frac{1}{\sqrt{8}}\right)^2 = \frac{4}{8} = \frac{1}{2}$

$y = \pm \frac{1}{\sqrt{2}}$.

$\underline{x = \frac{1}{\sqrt{8}}}$: $y^2 = 4x^2 = \frac{4}{8} = \frac{1}{2}$

$y = \pm \frac{1}{\sqrt{2}}$.

$\cdot$ Check:

$\cdot f\left(-\frac{1}{\sqrt{8}}, -\frac{1}{\sqrt{2}}\right) = 4 \cdot \frac{1}{8} - 6 \cdot \left(\frac{1}{\sqrt{16}}\right) + \frac{1}{2}$

$= 1 - \frac{6}{4} = -\frac{1}{2}$

$\cdot f\left(-\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{2}}\right) = 4 \cdot \frac{1}{8} + 6 \cdot \left(\frac{1}{\sqrt{16}}\right) + \frac{1}{2}$

$= 1 + \frac{6}{4} = \frac{5}{2}$

$\cdot f\left(\frac{1}{\sqrt{8}}, -\frac{1}{\sqrt{2}}\right) = \frac{5}{2}$

$\cdot f\left(\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

$\boxed{\text{Max val: } 5/2 \ \| \ \text{Min val: } -\tfrac{1}{2}}$

OVER →

3. Find the maximum and minimum values of $f(x,y) = xe^{y}$ subject to the constraint $x^2 + y^2 = 2$

- $f_x(x,y) = e^y$
- $f_y(x,y) = x \cdot \frac{\partial}{\partial y}[e^y] = xe^y$
- $g_x(x,y) = 2x$
- $g_y(x,y) = 2y$

$\underline{f_x = \lambda g_x}$: $e^y = \lambda 2x$

$\lambda = \frac{e^y}{2x}$

$\underline{f_y = \lambda g_y}$: $xe^y = \lambda 2y$

$\lambda = \frac{xe^y}{2y}$

$\frac{xe^y}{2y} = \frac{e^y}{2x}$

$2x^2 = 2y$

$y = x^2$

$\underline{g=k}$ $x^2 + y^2 = 2$

$y + y^2 = 2$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y = -2,\ y = 1.$

$\underline{y=-2:}$ $x^2 = -2$

No soln

$\underline{y=1:}$ $x^2 = 1$

$x = \pm 1$

- $f(-1,1) = -1 \cdot e^1 = -e$
- $f(1,1) = 1 \cdot e^1 = e$

$\boxed{\text{Max val: } e \quad \text{Min val: } -e}$

4. Find the maximum and minimum values of $f(x,y,z) = 2x + 6y + 3z$ subject to the constraint $x^2 + y^2 + z^2 = 49$. (Compare with 7.5 #15.)

- $f_x(x,y,z) = 2$ || $g_x = 2x$
- $f_y(x,y,z) = 6$ || $g_y = 2y$
- $f_z(x,y,z) = 3$ || $g_z = 2z$

- $\underline{f_x = \lambda g_x}$: $2 = \lambda 2x$

$1 = \lambda x \ \|\ 3 = 3\lambda x$

- $\underline{f_y = \lambda g_y}$: $6 = \lambda \cdot 2y$

$3 = \lambda y \ \|\ 3 = \lambda y$

- $\underline{f_z = \lambda g_z}$: $3 = \lambda 2z$

$\frac{3}{2} = \lambda z \ \|\ 3 = 2\lambda z$

$3\lambda x = \lambda y = 2\lambda z$

$3x = y = 2z$

$\underline{g=k}$: $x^2 + y^2 + z^2 = 49$

$\left(\frac{y}{3}\right)^2 + y^2 + \left(\frac{y}{2}\right)^2 = 49$

$\left(1 + \frac{1}{9} + \frac{1}{4}\right)y^2 = 49$

$\frac{49}{36} y^2 = 49$

$y^2 = 36$

$y = \pm 6$

$\underline{y = +6}$:

$3x = 6 \quad 2z = 6$

$x = 2 \quad z = 3$

$(x,y,z) = (2,6,3)$

$\underline{y=-6}$:

$3x = -6 \quad 2z = -6$

$x = -2 \quad z = -3$

$(x,y,z) = (-2,-6,-3)$

- $f(2,6,3) = 4 + 36 + 9 = 49$
- $f(-2,-6,-3) = -49$

OVER →

$\boxed{\text{Max val: } 49 \quad \text{Min val: } -49}$

5. (Compare with 7.5 #24, #25.)

(a) If x thousand dollars is spent on labor and y thousand dollars is spent on equipment, then the output at a certain factory is $Q(x,y) = \cancel{50}\,30\,x^{1/3}y^{2/3}$ units. If $\$10\cancel{0}2{,}000$ is available, how should this money be allocated between labor and equipment to generate the largest possible output?

• Want to maximize $Q(x,y)$ subject to $x+y=102$

• $Q_x(x,y) = 30 \cdot \frac{1}{3}x^{-2/3} \cdot y^{2/3} = 10x^{-2/3}y^{2/3}$

• $Q_y(x,y) = 30x^{1/3}\left(\frac{2}{3}y^{-\frac{1}{3}}\right) = 20x^{1/3}y^{-2/3}$

• $g_x(x,y) = 1$

• $g_y(x,y) = 1.$

$Q_x = \lambda g_x$: $10x^{-2/3}y^{2/3} = \lambda \cdot 1$

$Q_y = \lambda g_y$: $20x^{1/3}y^{-1/3} = \lambda \cdot 1$

$10x^{-2/3}y^{2/3} = 20x^{1/3}y^{-1/3}$

Mult both sides by $x^{2/3}y^{1/3}$:

$10y = 20x$

$y = 2x$

$g=k$: $x+y=102$

$3x = 102$

$x = 34$

$x=34$: $y = 2x = 68$

• $\boxed{\text{Output max. with \$34,000 labor and \$68,000 machines}}$

(b) Use the Lagrange multiplier λ to estimate the change in the maximum output of the factory if the available money increases by \$500. (You will need a calculator.)

• Note: $\$500 = \frac{1}{2} \cdot \$1000.$

• $\Delta \max \approx \lambda \cdot \frac{1}{2}$

• Must find λ for $(x,y) = (34,68)$:

• $\lambda = 10x^{-2/3}y^{2/3}$

• $\lambda = 10 \cdot (34)^{-2/3} \cdot (68)^{2/3}$

≈ 15.87

So $\Delta \max \approx \lambda \cdot \frac{1}{2}$

$\approx 15.87 \cdot \frac{1}{2}$

≈ 7.94

So the output will increase by about $\boxed{7.94}$ units.