Integration

1. $\int dx$

$$\int dx = x + C \text{ by power rule for antiderivatives}$$

2. $\int 2x^3 dx$

$$\begin{aligned}\int 2x^3 dx &= 2\int x^3 dx \\ &= 2(\frac{1}{4}x^4 + C_1) \text{ by power rule for antiderivatives} \\ &= \frac{1}{2}x^4 + C_2, \text{ where } C_2 = \frac{1}{2}C_1\end{aligned}$$

3. $\int (ax + bx^2)dx$

$$\begin{aligned}\int (ax + bx^2)dx &= a\int xdx + b\int x^2 dx \\ &= a(\frac{1}{2}x^2 + C_1) + b(\frac{1}{3}x^3 + C_2) \\ &= \frac{1}{2}ax^2 + \frac{1}{3}bx^3 + C_3, \text{ where } C_3 = aC_1 + bC_2\end{aligned}$$

4. $\int x^{-1}dx$

$$\begin{aligned}\int x^{-1}dx &= \int \frac{1}{x}dx \\ &= \ln x + C\end{aligned}$$

5. $\int \frac{x^2+1}{x} dx$

$$\begin{aligned}
\int \frac{x^2+1}{x} &= \int (x + \frac{1}{x}) dx \\
&= \int x dx + \int \frac{1}{x} dx \\
&= (\frac{1}{2}x^2 + C_1) + (\ln x + C_2) \\
&= \frac{1}{2}x^2 + \ln x + C_3, \text{ where } C_3 = C_1 + C_2
\end{aligned}$$

6. $\int_0^1 x dx$

$$\begin{aligned}
\int_0^1 x dx &= \frac{1}{2}x^2 \mid_0^1 \\
&= \frac{1}{2}(1^2 - 0^2) \\
&= \frac{1}{2}
\end{aligned}$$

7. $\int \frac{1}{2x+1} dx$

$$\begin{aligned}
\int \frac{1}{2x+1} dx &= \int \frac{1}{2} \cdot \frac{1}{u} du \text{ using } u = 2x+1,\ du = 2dx, \text{ so } dx = \frac{1}{2} du \\
&= \frac{1}{2} \int \frac{1}{u} du \\
&= \frac{1}{2}(\ln u + C_1) \\
&= \frac{1}{2} \ln u + C_2, \text{ where } C_2 = \frac{1}{2} C_1 \\
&= \frac{1}{2} \ln(2x+1) + C_2
\end{aligned}$$

8. $\int \frac{x}{3x-2}dx$

$$\begin{aligned}
\int \frac{x}{3x-2}dx &= \int \frac{\frac{u+2}{3}}{u}du \text{ using } u = 3x-2,\ x = \frac{u+2}{3},\ du = 3dx, \text{ and } dx = \frac{1}{3}du \\
&= \frac{1}{3}\int \frac{u+2}{u}du \\
&= \frac{1}{3}(\int du + \int \frac{2}{u}du) \\
&= \frac{1}{3}((u+C_1) + 2\int \frac{1}{u}du) \\
&= \frac{1}{3}((u+C_1) + 2(\ln u + C_2)) \\
&= \frac{1}{3}(u + 2\ln u + C_3), \text{ where } C_3 = C_1 + 2C_2 \\
&= \frac{1}{3}(u + 2\ln u) + C_4, \text{ where } C_4 = \frac{1}{3}C_3 \\
&= \frac{1}{3}(3x-2) + \frac{2}{3}\ln(3x-2) + C_4 \\
&= x + \frac{2}{3}\ln(3x-2) + C_5, \text{ where } C_5 = -\frac{2}{3} + C_4, \text{ since } \frac{1}{3}(3x-2) = x - \frac{2}{3}
\end{aligned}$$

9. $\int \frac{3x^2+1}{3x^3+3x}dx$

$$\begin{aligned}
\int \frac{3x^2+1}{3x^3+3x}dx &= \int \frac{1}{3}\cdot\frac{1}{u}du \ (u = 3x^3+3x,\ du = (9x^2+3)dx, \text{ and so } \frac{1}{3}du = (3x^2+1)dx) \\
&= \frac{1}{3}\int \frac{1}{u}du \\
&= \frac{1}{3}(\ln u + C_1) \\
&= \frac{1}{3}\ln u + C_2, \text{ where } C_2 = \frac{1}{3}C_1 \\
&= \frac{1}{3}\ln(3x^3+3x) + C_2 \\
&= \frac{1}{3}(\ln(3\cdot x\cdot(x^2+1)) + C_2 \\
&= \frac{1}{3}(\ln 3 + \ln x + \ln(x^2+1) + C_2 \text{ by the product rule for logarithms} \\
&= \frac{1}{3}\ln x + \frac{1}{3}\ln(x^2+1) + C_3, \text{ where } C_3 = \frac{1}{3}\ln 3 + C_2
\end{aligned}$$

10. $\int e^x dx$

$$\int e^x dx = e^x + C$$

11. $\int_0^{\ln 2} e^x dx$

$$\begin{aligned}\int_0^{\ln 2} e^x dx &= e^x \Big|_0^{\ln 2} \\ &= e^{\ln 2} - e^0 \\ &= 2 - 1 \text{ since } e^{\ln f(x)} = f(x) = \ln e^{f(x)} \text{ for any function } f(x) \\ &= 1\end{aligned}$$

12. $\int e^{3x} dx$

$$\begin{aligned}\int e^{3x} dx &= \int \frac{1}{3} \cdot e^u du \text{ using } u = 3x,\ du = 3dx, \text{ and } \frac{1}{3}du = dx \\ &= \frac{1}{3} \int e^u du \\ &= \frac{1}{3}(e^u + C_1) \\ &= \frac{1}{3}e^u + C_2 \text{ using } C_2 = \frac{1}{3}C_1 \\ &= \frac{1}{3}e^{3x} + C_2\end{aligned}$$

13. $\int_0^{\ln 2} e^{3x} dx$

$$\begin{aligned} \int_0^{\ln 2} e^{dx} dx &= \frac{1}{3} e^{3x} \,\Big|_0^{\ln 2} \text{ using \# 12} \\ &= \frac{1}{3}(e^{3 \cdot \ln 2} - e^{3 \cdot 0}) \\ &= \frac{1}{3}(e^{\ln 2^3} - 1) \\ &= \frac{1}{3}(8 - 1) \\ &= \frac{7}{3} \end{aligned}$$

14. $\int \frac{x^2 + xe^{2x}}{x} dx$

$$\begin{aligned} \int \frac{x^2 + xe^{2x}}{x} dx &= \int x dx + \int e^{2x} dx \\ &= \frac{1}{2} x + \frac{1}{2} e^{2x} + C \end{aligned}$$

15. $\int \frac{\ln x}{x} dx$

$$\begin{aligned} \int \frac{\ln x}{x} dx &= \int u du \text{ taking } u = \ln x \text{, and } du = \frac{1}{x} dx \\ &= \frac{1}{2} u^2 + C \\ &= \frac{1}{2} (\ln x)^2 + C \end{aligned}$$

16. $\int \frac{\ln x + 1}{x} dx$

$$\begin{aligned} \int \frac{\ln x + 1}{x} dx &= \int \frac{\ln x}{x} dx + \int x dx \\ &= \frac{1}{2} (\ln x)^2 + \frac{1}{2} x^2 + C \end{aligned}$$

17. $\int 2xe^{x^2}dx$

$$\begin{aligned}\int 2xe^{x^2}dx &= \int e^u du \text{ by using } u = x^2 \text{, so } du = 2xdx \\ &= e^u + C \\ &= e^{x^2} + C\end{aligned}$$

18. $\int (3x^2 + 2x)e^{x^3+x^2}dx$

$$\begin{aligned}\int (3x^2 + 2x)e^{x^3+x^2}dx &= \int e^u du \text{ by using } u = x^3 + x^2 \text{, so } du = (3x^2 + 2x)dx \\ &= e^u + C \\ &= e^{x^3+x^2} + C\end{aligned}$$

19. $\int 2xe^x dx$

$$\begin{aligned}\int 2xe^x dx &= 2\int xe^x \\ &= 2(xe^x - \int e^x) \text{ using integration by parts, with } u = x \text{, and } dv = e^x dx \\ &= 2(xe^x - e^x) + C\end{aligned}$$

20. $\int 2xe^{2x}dx$

Note that this is similar to $\int xe^x dx$, so we'll want to use integration by parts. The right choice of variables are

$$\begin{aligned} u &= 2x & dv &= e^{2x}dx \\ du &= 2dx & v &= \frac{1}{2}e^{2x}\end{aligned}$$

So by integration by parts,

$$\begin{aligned} \int 2xe^{2x}dx &= \int udv \\ &= uv - \int vdu \\ &= 2x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} \cdot 2dx \\ &= xe^{2x} - \int e^{2x}dx \\ &= xe^{2x} - \frac{1}{2}e^{2x} + C \end{aligned}$$

21. $\int x^2 e^x dx$

We want to use integration by parts here since there is no obvious candidate for u-substitution, and integration by parts gives us a way to decrease the exponent on the x^2 term, so our first choice is

$$\begin{aligned} u = x^2 &\qquad dv = e^x dx \\ du = 2xdx &\qquad v = e^x \end{aligned}$$

So we get that

$$\begin{aligned} \int x^2 e^x dx &= \int udv \\ &= uv - \int vdu \\ &= x^2 e^x - \int 2xe^x dx \\ &= x^2 e^x - 2\int xe^x dx \end{aligned}$$

Note that $\int xe^x dx$ is something we know how to calculate by integration by parts. In addition, we computed that in problem 19 (with $\cdot 2$), so

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2(xe^x - e^x) + C \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

22. $\int x^2 e^{x^2} dx$

This looks a bit ugly, especially with the x^2 term, so to ease the expression a bit, we shall use u-substitution, with $u = x^2$, to give us

$$\int x^3 e^{x^2} dx \quad = \quad \int \frac{1}{2} u e^u du$$

by noting that $du = 2xdx$, so $\frac{1}{2}du = xdx$, and splitting x^3 into $x^2 \cdot x$, so we can use $u = x^2$, and xdx for $\frac{1}{2}du$.

Now it is easy to see that we can use integration by parts here, and by observing that $\int \frac{1}{2} u e^u du = \frac{1}{2} e^u du$, we see that this is again very similar to problem 19, so we can calculate this easily:

$$\begin{aligned} \int x^3 e^{x^2} dx &= \int \frac{1}{2} u e^u du \\ &= \frac{1}{2} \int u e^u du \\ &= \frac{1}{2}(u e^u - e^u) + C \\ &= \frac{1}{2}(x^2 e^{x^2} - e^{x^2}) + C \end{aligned}$$

23. $\int_1^{e^2} \ln x dx$

We have no rule for calculating this, and it is clear that u-substitution is worthless in this situation, so we should apply integration by parts.

In fact, since we have a $\ln x$ here, it becomes a prime candidate for our choice of u since it's derivative is something more managable ($\frac{1}{x}$), so choosing

$$\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{1}{x} dx & v &= x, \end{aligned}$$

we get that

$$\begin{aligned}
\int_1^{e^2} \ln x dx &= \int_{x=1}^{x=e^2} u dv \\
&= uv \mid_{x=1}^{x=e^2} - \int_{x=1}^{x=e^2} v du \\
&= x \ln x \mid_1^{e^2} - \int_1^{e^2} x \cdot \frac{1}{2} dx \\
&= x \ln x \mid_1^{e^2} - \int_1^{e^2} dx \\
&= (x \ln x - x) \mid_1^{e^2} \\
&= (e^2 \ln e^2 - e^2) - (1 \cdot \ln 1 - 1) \\
&= (2e^2 - e^2) - (0 - 1) \\
&= e^2 + 1
\end{aligned}$$

24. $\int x \ln x dx$

Similar to problem 23, this is an integration by parts question, so we set

$$\begin{aligned}
u = \ln x \qquad & dv = x dx \\
du = \frac{1}{x} dx \qquad & v = \frac{1}{2} x^2,
\end{aligned}$$

which gives us that

$$\begin{aligned}
\int x \ln x dx &= \int u dv \\
&= uv - \int v du \\
&= \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \cdot \frac{1}{x} dx \\
&= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx \\
&= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \text{ (we calculated } \int x dx \text{ in problem 6)}
\end{aligned}$$

25. $\int x(3-x)^3 dx$

One way to approach this is to multiply everything out, and then integrate, but this is a cumbersome method with calculations that most of you probably would be better off not doing.

The best way to approach a question like this is to use integration by parts to reduce the exponent to eventually 0 (you may have to use it multiple times).

The best choice of u is the term with the smallest exponent, which here is the x term, so we set

$$\begin{aligned} u &= x & dv &= (3-x)^3 \\ du &= dx & v &= -\frac{1}{4}(3-x)^4 \end{aligned}$$

This gives us that

$$\begin{aligned}
\int x(3-x)^3 dx &= \int u dv \\
&= uv - \int v du \\
&= -\frac{1}{4}x(3-x)^4 + \frac{1}{4}\int (3-x)^4 dx \\
&= -\frac{1}{4}x(3-x)^4 - \frac{1}{4}\cdot\frac{1}{5}(3-x)^5 + C \\
& \quad \text{(you can calculate } \int (3-x)^4 dx \text{ using u-substitution)} \\
&= -\frac{1}{4}(3-x)^4(x + \frac{1}{5}(3-x)) + C \\
&= -\frac{1}{4}(3-x)^4(\frac{4}{5}x + \frac{3}{5}) + C \\
&= -\frac{1}{20}(3-x)^4(4x+3) + C
\end{aligned}$$

26. $\int (2x+3)^2(3x+5)^3 dx$

Again, this is integration by parts, and the smallest exponent is

2, on the $2x+3$ term, so we set

$$u = (2x+3)^2 \qquad dv = (3x+5)^3 dx$$
$$du = 4(2x+3)dx \qquad v = \frac{1}{12}(3x+5)^4$$

Be sure to remember to use chain rule when calculating du there, and to use a u-substitution to calculate the antiderivative of dv!

Now we get that

$$\begin{aligned}
\int (2x+3)^2(3x+5)^3 dx &= \int u dv \\
&= uv - \int v du \\
&= \frac{1}{12}(2x+3)^2(3x+5)^4 - \int \frac{1}{12}(3x+5)^4 \cdot 4(2x+3)dx \\
&= \frac{1}{12}(2x+3)^2(3x+5)^4 - \frac{1}{3}\int (3x+5)^4(2x+3)dx
\end{aligned}$$

Note that $\int (3x+5)^4(2x+3)dx$ has the exponent on the $3x+5$ term increased by 1, and the exponent on the $2x+3$ term decreased by 1. In general, this is always true in this type of situation, where the choice of u will have a power decreased by 1 and the choice for dv will have the power increased by 1.

In addition, this is an integral where we can apply integration by parts again, so we set

$$u = 2x+3 \qquad dv = (3x+5)^4 dx$$
$$du = 2dx \qquad v = \frac{1}{15}(3x+5)^5 dx$$

This gives us that

$$\begin{aligned}
\int (3x+5)^4(2x+3)dx &= \int udv \\
&= uv - \int vdu \\
&= \frac{1}{15}(2x+3)(3x+5)^5 - \int \frac{1}{15}(3x+5)^5 \cdot 2dx \\
&= \frac{1}{15}(2x+3)(3x+5)^5 - \frac{2}{15}\int (3x+5)^5 dx \\
&= \frac{1}{15}(2x+3)(3x+5)^5 - \frac{2}{15}\cdot\frac{1}{18}(3x+5)^6 + C \\
&\quad (\int (3x+5)^5 dx \text{ can be calculated using u-substitution})
\end{aligned}$$

So substituting this into the original calculation, we get that

$$\begin{aligned}
\int (2x+3)^2(3x+5)^3 dx &= \frac{1}{12}(2x+3)^2(3x+5)^4 - \frac{1}{3}\int (3x+5)^4(2x+3)dx \\
&= \frac{1}{12}(2x+3)^2(3x+5)^4 - \frac{1}{3}(\frac{1}{15}(2x+3)(3x+5)^5 - \\
&\quad \frac{2}{15}\cdot\frac{1}{18}(3x+5)^6) + K \text{ where K is some constant}
\end{aligned}$$

27. $\int \frac{e^{\frac{1}{x^2}}}{x^3} dx$

This is a u-substitution question.
Note that $(\frac{1}{x^2})' = -\frac{2}{x^3}$, so $u = \frac{1}{x^2}$ is an obvious choice.

This gives us that

$$\begin{aligned}
\int \frac{e^{\frac{1}{x^2}}}{x^3} dx &= \int -\frac{1}{2}e^u du \\
&= -\frac{1}{2}\int e^u du \\
&= -\frac{1}{2}e^u + C \\
&= -\frac{1}{2}e^{\frac{1}{x^2}} + C
\end{aligned}$$

28. $\int \frac{e^{\frac{1}{x^5}}}{x^6} dx$

Again, this is a u-substitution question, where the obvious choice of $u = \frac{1}{x^5}$, which gives us that

$$\begin{aligned}\int \frac{e^{\frac{1}{x^5}}}{x^6} dx &= \int -\frac{1}{5} e^u du \\ &= -\frac{1}{5} \int e^u du \\ &= -\frac{1}{5} e^u + C \\ &= -\frac{1}{5} e^{\frac{1}{x^5}} + C\end{aligned}$$

29. $\int \frac{1}{\sqrt{6x+5}} dx$

This is a simpler u-substitution question, with $u = 6x + 5$ a good choice, since $du = 6dx$, so

$$\begin{aligned}\int \frac{1}{\sqrt{6x+5}} dx &= \int \frac{1}{6} \cdot \frac{1}{\sqrt{u}} du \\ &= \frac{1}{6} \int u^{-\frac{1}{2}} du \\ &= \frac{1}{6} \cdot 2u^{\frac{1}{2}} + C \\ &= \frac{1}{3} \sqrt{6x+5} + C\end{aligned}$$

30. $\int \frac{(\sqrt{x}+1)^{\frac{1}{2}}}{\sqrt{x}} dx$

Note that $(\sqrt{x} + 1)' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, so $u = \sqrt{x} + 1$ is a good choice.

This gives us that

$$\begin{aligned}\int \frac{(\sqrt{x}+1)^{\frac{1}{2}}}{\sqrt{x}}dx &= \int 2u^{\frac{1}{2}}du\\ &= 2\int u^{\frac{1}{2}}du\\ &= 2\cdot\frac{2}{3}u^{\frac{3}{2}}+C\\ &= \frac{4}{3}(\sqrt{x}+1)^{\frac{3}{2}}+C\end{aligned}$$

Partial Derivatives

Calculate the second partials, including the mixed partials, of the following functions. In addition, evaluate which points are critical points or saddle points, and determine the relative extrema (relative maximum or minimum).

1. $f(x, y) = x + y$

   First, we must calculate the first partials in order to do anything for the problem, so

   $$f_x = 1 \quad \& \quad f_y = 1$$

   Note that f_x & f_y can never be 0, so there are no potential critical points. This isn't surprising since you can make $f(x, y)$ arbitrarily large by plugging in large numbers for x or y.

   Calculating the second partials, we get that

   $$\begin{aligned} f_{xx} &= 0 \quad \& \quad f_{yy} = 0 \\ f_{xy} &= 0 \end{aligned}$$

2. $f(x, y) = xy$

   Computing the first partials, we get that

   $$f_x = y \quad \& \quad f_y = x$$

   Setting $f_x = 0$ & $f_y = 0$, we get a critical point of (0, 0) for $f(x, y)$.

   Calculating the second partials, we get that

   $$\begin{aligned} f_{xx} &= 0 \quad \& \quad f_{yy} = 0 \\ f_{xy} &= 1 \end{aligned}$$

   This gives us that

   $$\begin{aligned} D(x, y) &= f_{xx}f_{yy} - (f_{xy})^2 \\ &= -1 \end{aligned}$$

   Note that $D(x, y)$ is always negative, so this means that (0, 0) is a saddle point.

3. $f(x, y) = x^2 + 2xy + y^2$

   For the first partials, we get

   $$f_x = 2x + 2y \quad \& \quad f_y = 2x + 2y$$

   Setting $f_x = 0$ and $f_y = 0$, we get that $x = -y$. This gives us an infinite set of critical points.

   Calculating the second partials, we get

   $$\begin{aligned} f_{xx} = 2 \quad &\& \quad f_{yy} = 2 \\ f_{xy} = 2, \end{aligned}$$

   giving us that

   $$\begin{aligned} D(x, y) &= f_{xx}f_{yy} - (f_{xy})^2 \\ &= 4 - 2^2 \\ &= 0, \end{aligned}$$

   so this test is inconclusive.

   However, note that $f(x, y) = (x + y)^2 \geq 0$, so when $x = -y$, the function is at its minimum.

   Along this line though, the surface remains at its minimum, so any point that satisfies $x = -y$ must be a saddle point.

4. $f(x, y) = e^{x^2+y^2}$

   For the first partials, we get

   $$f_x = e^{x^2+y^2} \cdot \frac{\partial}{\partial x}(x^2 + y^2) \qquad f_y = e^{x^2+y^2} \cdot \frac{\partial}{\partial y}(x^2 + y^2)$$
   $$f_x = 2xe^{x^2+y^2} \quad \& \quad f_y = 2ye^{x^2+y^2}$$

   Setting $f_x = 0$ & $f_y = 0$, we get a critical point of (0, 0), since $e^{x^2+y^2} > 0$ for any x & y.

Calculating the second partials, we get

$$\begin{aligned} f_{xx} &= 2e^{x^2+y^2} + 2x\frac{\partial}{\partial x}e^{x^2+y^2} \\ &= 2e^{x^2+y^2} + 4x^2e^{x^2+y^2} \\ &= (4x^2+2)e^{x^2+y^2} \end{aligned}$$

$$\begin{aligned} f_{yy} &= 2e^{x^2+y^2} + 2y\frac{\partial}{\partial y}e^{x^2+y^2} \\ &= 2e^{x^2+y^2} + 4y^2e^{x^2+y^2} \\ &= (4y^2+2)e^{x^2+y^2} \end{aligned}$$

$$\begin{aligned} f_{xy} &= 2xe^{x^2+y^2} \cdot \frac{\partial}{\partial y}(x^2+y^2) \\ &= 4xye^{x^2+y^2} \end{aligned}$$

This gives us that

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= (4x^2+2)(4y^2+2)e^{2(x^2+y^2)} - 4xye^{2(x^2+y^2)} \end{aligned}$$

Plugging in (0, 0), we get

$$\begin{aligned} D(0,0) &= (2)(2)e^{2\cdot 0} - 0 \\ &= 4 > 0, \end{aligned}$$

so (0, 0) is a relative maximum or minimum.

Checking f_{xx}, we get that

$$\begin{aligned} f_{xx}(0,0) &= 2e^0 \\ &= 2 > 0 \end{aligned}$$

Therefore (0, 0) is a relative minimum.

5. $f(x,y) = e^{xy}$

For the first partials, we get

$$f_x = e^{xy} \cdot \frac{\partial}{\partial x}(xy) \qquad f_y = e^{xy} \cdot \frac{\partial}{\partial y}(xy)$$

$$f_x = ye^{xy} \quad \& \quad f_y = xe^{xy}$$

Setting $f_x = 0$ & $f_y = 0$, we get that (0, 0) is the only critical point of f(x, y).

Calculating the second partials, we get

$$\begin{aligned} f_{xx} &= ye^{xy} \cdot \frac{\partial}{\partial x}(xy) \\ &= y^2 e^{xy} \end{aligned}$$

$$\begin{aligned} f_{yy} &= xe^{xy} \cdot \frac{\partial}{\partial y}(xy) \\ &= x^2 e^{xy} \end{aligned}$$

$$\begin{aligned} f_{xy} &= e^{xy} + ye^{xy} \cdot \frac{\partial}{\partial y}(xy) \\ &= e^{xy} + xye^{xy} \\ &= (1+xy)e^{xy} \end{aligned}$$

This gives us that

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= x^2y^2e^{2xy} - (1+xy)^2e^{2xy} \end{aligned}$$

Plugging in (0, 0), we get

$$\begin{aligned} D(0,0) &= 0 - (1+0)^2 \\ &= -1 < 0, \end{aligned}$$

so (0, 0) is a saddle point.

6. $f(x,y) = xye^{xy}$

   For the first partials, we get

$f_x = ye^{xy} + xy^2e^{xy}$ $\quad$ $f_y = xe^{xy} + x^2ye^{xy}$ by the product rule

$f_x = y(1+xy)e^{xy}$ & $f_y = x(1+xy)e^{xy}$

   Setting $f_x = 0$ and $f_y = 0$, we get that

$$y(1+xy)e^{xy} = 0 \quad \& \quad x(1+xy)e^{xy} = 0$$

   $\Rightarrow y(1+xy) = 0 \& x(1+xy) = 0$

   If $xy = -1$, then we get an infinite set of critical points of f(x, y).

   If $x = 0$ or $y = 0$, then $1 + xy = 1$, and so both must be simultaneously 0.

Therefore we get (0, 0) and $(x, -\frac{1}{x})$ as critical points of f(x, y), for all non-zero x.

Calculating the second partials, we get

$$\begin{aligned} f_{xx} &= y(ye^{xy} + (1+xy)ye^{xy}) \\ &= y^2(2+xy)e^{xy} \end{aligned}$$

$$\begin{aligned} f_{yy} &= x(xe^{xy} + (1+xy)xe^{xy}) \\ &= x^2(2+xy)e^{xy} \end{aligned}$$

$$\begin{aligned} f_{xy} &= (1+xy)e^{xy} + xye^{xy} + xy(1+xy)e^{xy} \\ &= (x^2y^2 + 3xy + 1)e^{xy} \end{aligned}$$

This gives us that

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= x^2xy^2(2+xy)e^{xy} - (x^2y^2 + 3xy + 1)e^{xy} \\ &= (x^3y^3 + x^2y^2 - 3xy - 1)e^{xy} \end{aligned}$$

Plugging in (0, 0) and $(x, -\frac{1}{x})$, we get

$$D(0,0) = -1 < 0$$

$$\begin{aligned} D(x, -\frac{1}{x}) &= (-1+1+1-1)e^{-1} \\ &= 0, \end{aligned}$$

so (0, 0) is a saddle point, and this test is indeterminate about $(x, -\frac{1}{x})$.

7. $f(x,y) = \ln(x+y)$

For the first partials, we get

$$f_x = \frac{1}{x+y} \quad \& \quad f_y = \frac{1}{x+y}$$

Notice that it is impossible for $f_x = 0$ and $f_y = 0$, so there are no critical points

Calculating the second partials, we get

$$\begin{aligned} f_{xx} &= -\frac{1}{(x+y)^2} \\ f_{yy} &= -\frac{1}{(x+y)^2} \\ f_{xy} &= -\frac{1}{(x+y)^2} \end{aligned}$$

8. $f(x, y) = x \ln(3x^2 y)$

Note that we can rewrite this as

$$\begin{aligned} x \ln(3x^2 y) &= x(\ln 3 + \ln x^2 + \ln y) \\ &= x \ln 3 + 2x \ln x + \ln y \end{aligned}$$

For the first partials, we then get

$$f_x = \ln 3 + (2 \ln x + 2x \cdot \frac{1}{x}) \qquad f_y = \frac{1}{y}$$

$$f_x = \ln 3 + 2 \ln x + 2 \quad \& \quad f_y = \frac{1}{y}$$

Note that $f_y = 0$ is impossible, so we have no critical points for f(x, y).

For the second partials, we get

$$\begin{aligned} f_{xx} &= \frac{2}{x} \\ f_{yy} &= -\frac{1}{y^2} \\ f_{xy} &= 0 \end{aligned}$$