1. *Find the equation of the tangent line to the curve* $4x^2y + y^3 = 16$ *at the point* $(1, 2)$.

To write down the equation of a line, we need its slope and some point on the line. We have a point, $(1, 2)$. To find the slope, we need to compute $\frac{dy}{dx}$. Differentiating both sides of the given equation, we get

$$\left(8xy + 4x^2\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

(Don't forget to treat y as a function of x; use the product rule to differentiate $4x^2y$ and the chain rule for y^3.) Solving for $\frac{dy}{dx}$,

$$\begin{aligned} & 4x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -8xy \\ \text{so} \qquad & \frac{dy}{dx}\left(4x^2 + 3y^2\right) = -8xy \\ \text{so} \qquad & \frac{dy}{dx} = \frac{-8xy}{4x^2 + 3y^2} \end{aligned}$$

To get the slope of the tangent line, plug in 1 for x and 2 for y. Now $\frac{dy}{dx}|_{x=1,y=2} = \frac{-16}{16} = -1$. Hence the equation of the line we're seeking is $y - 2 = -1(x - 1)$.

2. *A baker wishes to make a square pizza. He begins by producing* 450 *cubic centimeters of dough for the crust, and rolling it flat. The baker rolls out the dough in such a way that the top of the dough remains perfectly square at all times, and the thickness decreases at a rate of* 0.1 *centimeters per second. When the dough is* 2 *centimeters thick, how quickly is its width growing?*

   Recall the procedure for solving related rates problems:

   - Draw and label a picture. (Everything that changes over time should be labeled by a variable.)
   - Write down an equation relating all the variables.
   - Differentiate this equation with respect to time.
   - Plug in the given data.

   As time passes, the thickness of the dough decreases and its length and width increase. Hence, we need variables to represent each of these quantities. Let h denote the thickness and let s denote the length and width (since the dough is square, the length and width are equal). The key to solving this problem is to notice that the volume of the dough remains a constant 450 cubic centimeters throughout. Thus, $450 = s^2h$. Differentiating both sides,

$$0 = 2s\frac{ds}{dt}h + s^2\frac{dh}{dt}$$

   (Remember that both s and h are functions of time. When we differentiate in a related rates problem, we differentiate with respect to time. Also remember that to differentiate s^2h, we need to use the product rule.) Now we need to plug in. We're told that $h = 2$ and that $\frac{dh}{dt} = -0.1$. (Since h is decreasing, its derivative is negative.) We want to solve for $\frac{ds}{dt}$. We need a value for s; plugging $h = 2$ into the equation $450 = s^2h$, we find that $s = 15$. Plugging everything into the derivative, we have

$$0 = 2 \cdot 15 \cdot \frac{ds}{dt} \cdot 2 + 15^2 \cdot (-0.1)$$

   Solving for $\frac{ds}{dt}$, we get $\frac{ds}{dt} = \frac{9}{24}$. Thus the length of the dough changes at a rate of $\frac{9}{24}$ centimeters per second.

3. *Let* $f(x) = 2x^3 + 5x^2 - 4x + 3$.

   (a) *Where is* $f(x)$ *concave up? Where is it concave down?*

   To determine the concavity of a function, we need to compute its second derivative.

$$f'(x) = 6x^2 + 10x - 4, \qquad \text{and}$$
$$f''(x) = 12x + 10$$

   Note that $f''(x) = 0$ only when $x = -\frac{5}{6}$. This point splits the real line into two intervals: $(-\infty, -\frac{5}{6})$ and $(-\frac{5}{6}, \infty)$. On each of these intervals, $f''(x)$ is either positive or negative throughout. To find out which, we choose a test point from each interval, and plug it into $f''(x)$:

   - $f''(-1) = -12 + 10 = -2 < 0$, so $f(x)$ is concave down on $(-\infty, -\frac{5}{6})$.
   - $f''(0) = 0 + 10 = 10 > 0$, so $f(x)$ is concave up on $(-\frac{5}{6}, \infty)$.

   (b) *Find the critical points of* $f(x)$ *and use the second derivative test to classify them as relative maxima or relative minima.*

   The critical points of $f(x)$ are the places where $f'(x)$ is zero or undefined. Recall that we computed $f'(x)$ above. Factoring, $f'(x) = (3x - 1)(2x + 4)$. $f'(x)$ is never undefined, and is zero when $x = \frac{1}{3}$ and when $x = 2$. Thus the critical numbers of f are $\frac{1}{3}$ and -2; the critical points (i.e. the points corresponding to the critical numbers) are $(\frac{1}{3}, \frac{62}{27})$ and $(-2, 15)$. To apply the second derivative test, we take the critical numbers and plug them into $f''(x)$:

   - $f''(\frac{1}{3}) = 4 + 10 = 14 > 0$, so $(\frac{1}{3}, \frac{62}{27})$ is a relative minimum.
   - $f''(-2) = -24 + 10 = -14 < 0$, so $(-2, 15)$ is a relative maximum.

4. (a) Find all asymptotes of the curve $f(x) = \dfrac{2x^2 - 9x + 4}{x^2 - 3x - 4}$.

We'll start with vertical asymptotes. The first step is to find all values of x for which the denominator of $f(x)$ is zero; these are the places where we *might* have vertical asymptotes. Note that $x^2 - 3x - 4 = (x+1)(x-4)$; it's zero when $x = -1$ and when $x = 4$. To check whether we actually have asymptotes there, we need to compute some limits:

- $\displaystyle\lim_{x\to -1^+} \frac{2x^2 - 9x + 4}{x^2 - 3x - 4} = \lim_{x\to -1^+} \frac{(2x-1)(x-4)}{(x+1)(x-4)} = \lim_{x\to -1^+} \frac{2x-1}{x+1} = -\infty$, so we have a vertical asymptote at $x = -1$. (For the last step in the computation of the limit, note that the numerator tends to -1, while the denominator tends to 0. Since x approaches -1 from the right, the denominator tends to 0 from above, so the fraction is negative.)
- $\displaystyle\lim_{x\to 4^+} \frac{2x^2 - 9x + 4}{x^2 - 3x - 4} = \lim_{x\to 4^+} \frac{(2x-1)(x-4)}{(x+1)(x-4)} = \lim_{x\to 4^+} \frac{2x-1}{x+1} = \frac{7}{5}$, so we *do not* have a vertical asymptote at $x = 4$. (If you're curious, this means we just have a "hole" in the graph at $(4, \frac{7}{5})$.)

Now let's check for horizontal asymptotes. To do this, we compute more limits:

- $\displaystyle\lim_{x\to\infty} \frac{2x^2 - 9x + 4}{x^2 - 3x - 4} = \lim_{x\to\infty} \frac{2 - \frac{9}{x} + \frac{4}{x^2}}{1 - \frac{3}{x} - \frac{4}{x^2}} = 2$, so we have a horizontal asymptote at $y = 2$.
- Likewise, $\displaystyle\lim_{x\to -\infty} \frac{2x^2 - 9x + 4}{x^2 - 3x - 4} = 2$, which further confirms that we have a horizontal asymptote at $y = 2$. (When there are asymptotes, these two limits will usually have the same value...but not always. You do need to check both. Some functions have two different horizontal asymptotes.)

(b) *Sketch a graph of some function $g(t)$ for which:*

- *$g'(t)$ is positive on $(-\infty, -4)$ and $(4, \infty)$, but negative on $(-4, 4)$;*
- *$g''(t)$ is positive on $(-2, 0)$, but negative on $(-\infty, -2)$, $(0, 4)$, and $(4, \infty)$.*

(Sorry; I couldn't find a scanner. Stop by my office hours and I can draw the answer for you. The main "trick" to this question is that the graph should have either a vertical asymptote or a sharp point at $x = 4$; that's what lets you go from decreasing to increasing while remaining concave down.)

5. *McGraw-Hill, a textbook publisher, wants to determine a price for the latest edition of their most popular calculus textbook. Their research indicates that a book priced at p dollars will sell $D(p) = 1500000p^{-1} - 1000$ copies. Naturally, McGraw-Hill intends to produce only as many textbooks as they can sell. Textbook production incurs a flat overhead cost of $80,000, as well as a material cost of $15 per book. At what price should McGraw-Hill sell their calculus book in order to maximize their profits?*

   Note: the numbers get a bit large here – sorry about that.

   Start by getting a formula for the thing you want to minimize or maximize (in this case, profit). We know that $P(p) = R(p) - C(p)$, where $P(p)$ indicates profit, $R(p)$ indicates revenue, and $C(p)$ indicates cost. Let's get formulas for $R(p)$ and $C(p)$:

   - $R(p) = p \cdot D(p) = 1500000 - 1000p$, since we sell $D(p)$ books for p dollars each.
   - $C(p) = [\text{overhead cost}] + [\text{material cost}] = 80000 + 15 \cdot D(p) = 80000 + 22500000p^{-1} - 15000$.

   Thus,

$$P(p) = R(p) - C(p) = 1500000 - 1000p - 80000 - 22500000p^{-1} + 15000$$

   We want to find the absolute maximum of this function. (Note that, unlike in many problems, there is no clear interval for p: certainly $p \geq 0$, but there's no upper limit to what we might want to charge.) Start by finding the critical numbers for $P(p)$:

$$P'(p) = -1000 + 22500000p^{-2}$$

   so $P'(p) = 0$ when $22500000p^{-2} = 1000$, i.e. $22500p^{-2} = 1$, i.e. $22500 = p^2$, i.e. $p = 150$. Since $p = 150$ is the *only* critical number of P, we know that if the second derivative test says it's a relative maximum, then in fact it must be the *absolute* maximum. (This *only* works when there's only one critical number, and *only* when the second derivative test applies. You should only use this test when there's no interval to work with; otherwise you should do things the "normal" way.) Note that

$$P''(p) = -45000000p^{-3}$$

   To apply the second derivative test, we need to compute $P''(150)$. The numbers could get messy, but it's clear that we'll get a negative result, which is all that matters. Since $P''(150) < 0$, $p = 150$ is a relative maximum; as discussed above, it must be an absolute maximum. Thus McGraw-Hill should sell their textbook for $150 per copy.

6. *Solve for x:*

   (a) $2^{8-x} = 4^x$

   The key is to rewrite the equation so that both of the exponential functions have the same base. Here, since $4 = 2^2$, we have $2^{8-x} = (2^2)^x$, so $2^{8-x} = 2^{2x}$. Now we see that $8 - x = 2x$, so $8 = 3x$, and so $x = 8/3$.

   (b) $2^{3x} = 24 \cdot 3^{-x}$

   This time we can't get both exponential functions to have the same base, so let's get them both to have the same exponent. We also need to get them both on the same side of the equation. Multiplying through by 3^x, we have $2^{3x}3^x = 24$. Note that $2^{3x} = (2^3)^x = 8^x$, so $8^x 3^x = 24$. Since the exponential functions have the same exponent, we can combine them: now $24^x = 24$, so clearly $x = 1$.

   (c) $\ln x - \ln 4 = 2(\ln 3 - \ln 2)$

   We need to isolate $\ln x$, then simplify the logarithms as much as possible.

$$\begin{aligned}
\ln x &= 2(\ln 3 - \ln 2) + \ln 4 \\
&= 2 \ln \frac{3}{2} + \ln 4 \\
&= \ln \left(\frac{3}{2}\right)^2 + \ln 4 \\
&= \ln \frac{9}{4} + \ln 4 \\
&= \ln 9
\end{aligned}$$

   Hence $x = 9$.