Directions:

1. Write your name with one character in each box below.
2. Show all work. No credit for answers without work.
3. You are permitted to use one 8.5 inch by 11 inch sheet of prepared notes. No other aides are allowed.

1. **[15 points]** Determine whether the following vectors are linearly independent. If the vectors are linearly dependent, give a dependence relation.

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ -9 \\ -6 \\ 9 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ -2 \end{bmatrix}$$

2. **[10 points]** Characterize when the following vectors are linearly dependent in terms of simple conditions on h and k.

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ h \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ k \\ -h \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \\ k \\ 0 \end{bmatrix}$$

3. Transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

   (a) **[9 points]** Let T_1 be the transform given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ x_2 + 1 \end{bmatrix}$. Is T_1 one-to-one/injective? Is T_1 onto/surjective? Is T_1 linear? Explain.

   (b) **[6 points]** Let T_2 be the transform given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$ and let T_3 be the transformation that rotates points by $\pi/6$ radians. Find the standard matrix for the composition transformation $T_2 \circ T_3$ given by $\mathbf{x} \mapsto T_2(T_3(\mathbf{x}))$.

4. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transform, and let A be the standard matrix for T.

   (a) **[2 points]** How many rows does A have? How many columns?

   (b) **[8 points]** By analyzing the pivot positions of A, prove that if $n > m$ then T is not one-to-one/injective.

5. **[20 points]** Find the inverse of the following matrix.

$$\begin{bmatrix} -8 & 1 & -9 \\ 3 & 0 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

6. **[10 points]** Find elementary matrices E_1, E_2, E_3 such that $E_3E_2E_1A = B$.

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 1 & 2 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

7. **[10 parts, 2 points each]** True/False. Assume the matrix operations below are well-defined. Justify your answers.

   (a) Every elementary matrix is square.

   (b) $(A+B)C = AC + BC$

   (c) $(A+B)(A+B) = A^2 + 2AB + B^2$

   (d) If $AB = BA$, then A and B are inverses of one another.

   (e) If A and B are invertible $(n \times n)$-matrices, then A and B are row-equivalent.

   (f) If A and B are invertible $(n \times n)$-matrices, then $A + B$ is also invertible.

   (g) If A and B are invertible $(n \times n)$-matrices, then AB is also invertible.

   (h) An $(n \times n)$-matrix A is invertible if and only if its transpose A^T is invertible.

   (i) If S and T are linear transforms from $\mathbb{R}^n$ to $\mathbb{R}^n$ and S and T are equal on at least n points in $\mathbb{R}^n$, then $S = T$.

   (j) If T is a linear transform and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is also linearly independent.