Directions:

1. Write your name with one character in each box below.
2. Show all work. No credit for answers without work.

1. **[15 points]** Solve the following system of equations.

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & - & x_3 & = & 0 \\ -30x_1 & + & 36x_2 & - & 7x_3 & = & 7 \\ -5x_1 & + & 6x_2 & - & x_3 & = & 1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -30 & 36 & -7 & 7 \\ -5 & 6 & -1 & 1 \end{bmatrix} \underset{\sim}{\overset{R2 \overset{+}{=} 30(R1)}{R3 \overset{+}{=} 5(R1)}} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 6 & -37 & 7 \\ 0 & 1 & -6 & 1 \end{bmatrix} \overset{R2 \leftrightarrow R3}{\sim} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -6 & 1 \\ 0 & 6 & -37 & 7 \end{bmatrix} \overset{R3 \overset{+}{=} (-6)R2}{\sim} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\overset{R3 \overset{\times}{=} -1}{\sim} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \underset{\sim}{\overset{R1 \overset{+}{=} R3}{R2 \overset{+}{=} 6R3}} \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \overset{R1 \overset{+}{=} R2}{\sim} \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

So the unique soln is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \\ -1 \end{bmatrix}$.

2. **[15 points]** Give an equation for the components of **b** that determines when the system $A\mathbf{x} = \mathbf{b}$ is consistent.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 7 \\ -2 & -2 & -14 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & b_1 \\ 1 & 1 & 7 & b_2 \\ -2 & -2 & -14 & b_3 \end{bmatrix} \overset{R1 \leftrightarrow R2}{\sim} \begin{bmatrix} 1 & 1 & 7 & b_2 \\ 0 & 1 & 2 & b_1 \\ -2 & -2 & -14 & b_3 \end{bmatrix} \overset{R3 \overset{+}{=} 2R1}{\sim} \begin{bmatrix} 1 & 1 & 7 & b_2 \\ 0 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_3 + 2b_2 \end{bmatrix}$$

The system $A\vec{x} = \vec{b}$ is consistent iff the last column is not a pivot column.
This is the case if and only if $\boxed{2b_2 + b_3 = 0}$.

3. **[15 points]** List the elementary row operations and give a brief description of each.

Replacement: Replace Row i with (Row i) + (a scalar multiple of some other row)

Scaling: Multiply Row i by a nonzero constant

Exchange: Swap two rows

4. **[15 points]** Determine which values of h, if any, make the linear system represented by the following augmented matrix have infinitely many solutions.

$$\begin{bmatrix} 3 & -2 & 5 & 1 \\ 2 & h & 2 & -1 \\ 1 & -1 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix} 1 & -1 & 3 & 2 \\ 2 & h & 2 & -1 \\ 3 & -2 & 5 & 1 \end{bmatrix} \xrightarrow[R3 \pm (-3)R1]{R2 \pm (-2)R1} \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & h+2 & -4 & -5 \\ 0 & 1 & -4 & -5 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -4 & -5 \\ 0 & h+2 & -4 & -5 \end{bmatrix}$$

$$\xrightarrow{R3 \pm -(h+2)R2} \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & -4+4(h+2) & -5+5(h+2) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 4(h+1) & 5(h+1) \end{bmatrix}$$

If $h=-1$, then the last row is all zeros and the pivot columns are the first two columns. Since the last column is not a pivot, this is a consistent system. Since the 3rd column is also not a pivot, x_3 is a free variable and the system has infinitely many solutions. If $h \neq -1$, then the first 3 columns are pivot columns and the last is not, indicating the system has a unique soln. Therefore the only value for h that gives infinitely many solns is $\boxed{h=-1}$.

5. **[15 points]** Given the augmented matrix, express the solution set in parametric form.

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & RHS \end{array}$$
$$\left[\begin{array}{ccccccc|c} 1 & 5 & 0 & 0 & -7 & 4 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Already in reduced echelon form. Free: x_2, x_5, x_6

$$x = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{bmatrix} -5x_2 + 7x_5 - 4x_6 + 1 \\ x_2 \\ -2x_6 - 1 \\ -6x_5 + x_6 \\ x_5 \\ x_6 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ 0 \\ 0 \\ -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_2, x_5, x_6 \in \mathbb{R}.$$

6. **[5 parts, 2 points each]** True/False. Justify your answers.

   (a) Every matrix is row-equivalent to infinitely many matrices.

   (b) Every inconsistent linear system can be made consistent by deleting a carefully chosen equation.

   (c) If $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{c}$ are consistent systems, then the solution sets are translations of one another.

   (d) If $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ and $n \geq m$, then $\mathrm{Span}(\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}) = \mathbb{R}^m$.

   (e) If $\mathbf{u}$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{v}$ is a scalar multiple of $\mathbf{b}$, then $\mathbf{u} + \mathbf{v}$ is a scalar multiple of $\mathbf{a} + \mathbf{b}$.

(a): FALSE. A matrix with all zero entries is only row-equivalent to itself.

(b) FALSE. For example,
$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{aligned}$$
remains inconsistent after removing any single equation.

(c) True. Since $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ are both consistent, each solution set can be translated to the solution set of the homogeneous system $A\vec{x} = \vec{0}$ and hence they can be translated to each other.

(d) FALSE. If $\vec{a}_1 = \cdots = \vec{a}_n = \vec{0}_m$ Then $\mathrm{span}(\{\vec{a}_1, \ldots, \vec{a}_n\}) = \{\vec{0}_m\} \neq \mathbb{R}^m$.

(e) FALSE: Suppose $\vec{u} = (1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = (2)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Now $\vec{u} + \vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{a} + \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{u} + \vec{v}$ is not a scalar multiple $\vec{a} + \vec{b}$.

7. An economy has 3 sectors: tech, food, and energy. The output of the tech sector is consumed as follows: 4/5 to tech, 1/10 to food, and 1/10 to energy. The output of the food sector is consumed as follows: 1/3 to tech, 1/6 to food, and 1/2 to energy. The output of the energy sector is consumed as follows: 1/4 to tech, 1/4 to food, and 1/2 to energy.

   (a) **[10 points]** Let p_t, p_f, and p_e, be the total cost (equivalently, the total value) in billions of the food, energy, and tech sectors, respectively. Assuming that the total cost (or total value) of each sector equals the that sector's total expenses, obtain a linear system in variables p_t, p_f, and p_e.

	tech	food	energy
tech	4/5	$\frac{1}{3}$	$\frac{1}{4}$
food	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$
energy	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{2}$

$$\boxed{\begin{aligned} \tfrac{4}{5}p_t + \tfrac{1}{3}p_f + \tfrac{1}{4}p_e &= p_t \\ \tfrac{1}{10}p_t + \tfrac{1}{6}p_f + \tfrac{1}{4}p_e &= p_f \\ \tfrac{1}{10}p_t + \tfrac{1}{2}p_f + \tfrac{1}{2}p_e &= p_e \end{aligned}}$$

or

$$\boxed{\begin{bmatrix} -\frac{1}{5} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{10} & -\frac{5}{6} & \frac{1}{4} \\ \frac{1}{10} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \vec{p} = \vec{0}}$$

   (b) **[5 points]** Given that the total value of the economy is \$120 billion (i.e. $p_t + p_f + p_e = 120$), find the value of each sector. (Hint: first scale each row to eliminate fractions and then try to avoid reintroducing them.)

$$\begin{bmatrix} -\frac{1}{5} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{10} & -\frac{5}{6} & \frac{1}{4} \\ \frac{1}{10} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \xrightarrow{\substack{R1 \doteq 60 \\ R2 \doteq 60 \\ R3 \doteq 10}} \begin{bmatrix} -12 & 20 & 15 \\ 6 & -50 & 15 \\ 1 & 5 & -5 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix} 1 & 5 & -5 \\ 6 & -50 & 15 \\ -12 & 20 & 15 \end{bmatrix} \xrightarrow{\substack{R2 \doteq (-6)R1 \\ R3 \doteq (12)R1}} \begin{bmatrix} 1 & 5 & -5 \\ 0 & -80 & 45 \\ 0 & 80 & -45 \end{bmatrix}$$

$$\xrightarrow{R3 \doteq R2} \begin{bmatrix} 1 & 5 & -5 \\ 0 & -80 & 45 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \doteq (-\frac{1}{5})} \begin{array}{c} \begin{array}{ccc} p_t & p_f & p_e \end{array} \\ \begin{bmatrix} 1 & 5 & -5 \\ 0 & 16 & -9 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

x_3 free

$16x_2 = 9x_3$; $x_2 = \frac{9}{16}x_3$

$x_1 = -5x_2 + 5x_3 = \frac{-45}{16}x_3 + 5x_3 = \frac{35}{16}x_3$

So the gen soln is $\vec{p} = \begin{bmatrix} \frac{35}{16}x_3 \\ \frac{9}{16}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{35}{16} \\ \frac{9}{16} \\ 1 \end{bmatrix}$.

We know $\frac{35}{16}x_3 + \frac{9}{16}x_3 + x_3 = 120$

$\frac{x_3}{16}(35 + 9 + 16) = 120$

$\frac{x_3}{16}(60) = 120 \Rightarrow x_3 = 32.$

So $\vec{p} = 32 \begin{bmatrix} \frac{35}{16} \\ \frac{9}{16} \\ 1 \end{bmatrix} = \begin{bmatrix} 70 \\ 18 \\ 32 \end{bmatrix}$.

$\boxed{\text{Tech worth \$70 billion, food worth \$18 billion, energy worth \$32 billion.}}$