Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[3 parts, 1 point each]** True/False. Justify your answers.

(a) There is a vector $\mathbf{u} \in \mathbb{R}^n$ such that for each vector $\mathbf{v} \in \mathbb{R}^n$, we have $\mathbf{u} \in \text{Span}\{\mathbf{v}\}$.

$\boxed{\text{True}}$; Let $\vec{u} = \vec{0}_n$. We have $\vec{u} \in \text{Span}(\vec{v})$ for each $\vec{v} \in \mathbb{R}^n$.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are both linear combinations of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_p$, then so is $\mathbf{u} + \mathbf{v}$.

$\boxed{\text{True}}$. If $\vec{u} = \alpha_1 \vec{a}_1 + \cdots + \alpha_p \vec{a}_p$ and $\vec{v} = \beta_1 \vec{a}_1 + \cdots + \beta_p \vec{a}_p$, then

$\vec{u} + \vec{v} = (\alpha_1 \vec{a}_1 + \cdots + \alpha_p \vec{a}_p) + (\beta_1 \vec{a}_1 + \cdots + \beta_p \vec{a}_p) = (\alpha_1 + \beta_1)\vec{a}_1 + \cdots + (\alpha_p + \beta_p)\vec{a}_p$.

(c) For all vectors $\mathbf{u}$ and $\mathbf{v}$, we have that $\text{Span}\{\mathbf{u}\} = \text{Span}\{\mathbf{v}\}$ implies $\mathbf{u} = \mathbf{v}$.

$\boxed{\text{FALSE}}$. If $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, then $\text{Span}\{\vec{u}\} = \text{Span}\{\vec{v}\} = \left\{ \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$

but $\vec{u} \neq \vec{v}$.

2. **[1 point]** Determine if $\mathbf{b}$ is in the span of $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ where $\mathbf{a_1} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, $\mathbf{a_3} = \begin{bmatrix} 1 \\ 6 \\ 19 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ -3 & 4 & 6 & 0 \\ 2 & 5 & 19 & 0 \end{bmatrix} \xrightarrow{R1 \div -1} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -3 & 4 & 6 & 0 \\ 2 & 5 & 19 & 0 \end{bmatrix} \xrightarrow[R3 \pm (-2)R1]{R2 \pm (3)R1} \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 3 & -3 \\ 0 & 7 & 21 & 2 \end{bmatrix}$$

$$\xrightarrow{R3 \pm (-7)R2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 23 \end{bmatrix}$$

Since the last column is a pivot column, the system is inconsistent and therefore $\vec{b}$ is $\boxed{\text{not in the span}}$ of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

3. **[2 parts, 3 points each]** Given A and $\mathbf{b}$ below, solve for $\mathbf{x}$ in the matrix equation $A\mathbf{x} = \mathbf{b}$.

(a) $A = \begin{bmatrix} 5 & 3 & -1 \\ 0 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 31 \\ -8 \\ -15 \end{bmatrix}$.

$$\begin{bmatrix} 5 & 3 & -1 & 31 \\ 0 & 1 & -2 & -8 \\ -2 & 1 & 1 & -15 \end{bmatrix} \xrightarrow{R1 \overset{+}{=} (2)R3} \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & -2 & -8 \\ -2 & 1 & 1 & -15 \end{bmatrix} \xrightarrow{R3 \overset{+}{=} 2R1} \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & -2 & -8 \\ 0 & 11 & 3 & -13 \end{bmatrix}$$

$$\xrightarrow{R3 \overset{+}{=} (-11)R2} \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 25 & 75 \end{bmatrix} \xrightarrow{R3 \doteq \frac{1}{25}} \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[R2 \overset{+}{=} 2R3]{R1 \overset{+}{=} (-1)R3} \begin{bmatrix} 1 & 5 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R1 \overset{+}{=} (-5)R2} \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\boxed{\text{So the unique soln is } \vec{x} = \begin{bmatrix} 8 \\ -2 \\ 3 \end{bmatrix}.}$$

(b) $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 2 \\ 3 & -1 & -1 \\ 4 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 23 \\ 13 \\ 11 \\ 5 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -1 & 2 & 23 \\ 1 & 0 & 2 & 13 \\ 3 & -1 & -1 & 11 \\ 4 & 1 & 0 & 5 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 0 & 2 & 13 \\ 2 & -1 & 2 & 23 \\ 3 & -1 & -1 & 11 \\ 4 & 1 & 0 & 5 \end{bmatrix} \xrightarrow[\substack{R3 \overset{+}{=} (-3)R1 \\ R4 \overset{+}{=} (-4)R1}]{R2 \overset{+}{=} (-2)R1} \begin{bmatrix} 1 & 0 & 2 & 13 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & -7 & -28 \\ 0 & 1 & -8 & -47 \end{bmatrix}$$

$$\xrightarrow{R2 \doteq -1} \begin{bmatrix} 1 & 0 & 2 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -7 & -28 \\ 0 & 1 & -8 & -47 \end{bmatrix} \xrightarrow[R3 \overset{+}{=} (-1)R2]{R3 \overset{+}{=} R2} \begin{bmatrix} 1 & 0 & 2 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -25 \\ 0 & 0 & -10 & -50 \end{bmatrix} \xrightarrow[R4 \doteq -\frac{1}{10}]{R3 \doteq -\frac{1}{5}} \begin{bmatrix} 1 & 0 & 2 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow[\substack{R2 \overset{+}{=} (-2)R3 \\ R4 \overset{+}{=} (-1)R3}]{R1 \overset{+}{=} (-2)R3} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the unique soln is $\vec{x} = \begin{bmatrix} 3 \\ -7 \\ 5 \end{bmatrix}$.