Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[6 points]** Find the general solution of the system with the following augmented matrix.

$$\begin{bmatrix} 3 & 0 & -3 & 6 & -2 & -3 \\ 3 & 0 & -3 & 6 & -3 & -6 \\ 10 & 1 & -6 & 21 & -3 & 3 \\ -8 & 0 & 8 & -16 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{R2 \overset{\times}{=} \frac{1}{3}} \begin{bmatrix} 3 & 0 & -3 & 6 & -2 & -3 \\ 1 & 0 & -1 & 2 & -1 & -2 \\ 10 & 1 & -6 & 21 & -3 & 3 \\ -8 & 0 & 8 & -16 & 3 & 1 \end{bmatrix} \xrightarrow[R4 \overset{+}{=} 8R2]{R1 \overset{+}{=} (-3)R2,\ R3 \overset{+}{=} (-10)R2} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 3 \\ 1 & 0 & -1 & 2 & -1 & -2 \\ 0 & 1 & 4 & 1 & 7 & 23 \\ 0 & 0 & 0 & 0 & -5 & -15 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 0 & -1 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 4 & 1 & 7 & 23 \\ 0 & 0 & 0 & 0 & -5 & -15 \end{bmatrix}$$

$$\xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 0 & -1 & 2 & -1 & -2 \\ 0 & 1 & 4 & 1 & 7 & 23 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & -5 & -15 \end{bmatrix} \xrightarrow{R4 \overset{+}{=} (5)R3} \begin{bmatrix} 1 & 0 & -1 & 2 & -1 & -2 \\ 0 & 1 & 4 & 1 & 7 & 23 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R2 \overset{+}{=} (-7)R3]{R1 \overset{+}{=} R3} \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 & 2 & 0 & 1 \\ 0 & 1 & 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} \text{pivot} & \text{pivot} & & & \text{pivot} & \\ \text{basic} & \text{basic} & \text{free} & \text{free} & \text{basic} & \end{array} \end{array}$$

So the general soln is given by

$$\boxed{\begin{aligned} x_5 &= 3 \\ x_4 &\text{ free} \\ x_3 &\text{ free} \\ x_2 &= -4x_3 - x_4 + 2 \\ x_1 &= x_3 - 2x_4 + 1 \end{aligned}}$$

OR

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 + x_3 - 2x_4 \\ 2 - 4x_3 - x_4 \\ x_3 \\ x_4 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} + a \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad a, b \in \mathbb{R}}$$

2. **[2 parts, 1.5 points each]** Let A be a (8×4)-matrix such that each of the four columns is a pivot column.

   (a) Give the *reduced row-echelon form* of A.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (b) Describe the *row-echelon form(s)* of A.

These are the matrices of the form

$$\begin{array}{c} \\ \\ \\ \\ 4 \end{array} \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \\ & & 0 & \end{bmatrix}$$

where $\underline{a, e, h, j}$ are non-zero entries and $\underline{b, c, d, f, g, i}$ are arbitrary entries

3. **[1 point]** Let A be an $(m \times n)$ augmented matrix that represents a linear system with a unique solution. What can be said about the relationship between m and n?

By the existence and uniqueness theorem, the first $n-1$ columns are pivot columns and the last column is a non-pivot column.

Having $n-1$ pivot columns requires at least $n-1$ rows, and so $m \geq n-1$.

Conversely if $m \geq n-1$, then the matrix might be

$$\begin{array}{c} n-1 \\ \\ m-(n-1) \end{array} \begin{bmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ & & & 0 \\ & 0 & & \vdots \\ & & & 0 \end{bmatrix}$$

in which the only solution is to set all variables equal to zero. So we can say $\boxed{m \geq n-1}$.