Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Use Cayley's Formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees.

2. Let G be a graph with m edges and maximum degree k, where $k \geq 3$.

   (a) Let M be a maximum matching in G. Prove that the number of edges joining vertices saturated by M to vertices not saturated by M is at most $(k-1)|M|$.

   (b) Prove that $\alpha'(G) \geq 2m/(3k-1)$.

3. A *doubly stochastic matrix* Q is a nonnegative real matrix in which every row and every column sums to 1. Prove that a doubly stochastic matrix Q can be expressed as $Q = c_1P_1 + \cdots + c_mP_m$ where $c_1, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, \ldots, P_m$ are permutation matrices. For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   Hint: Use induction on the number of nonzero entries in Q.

4. Determine the maximum number of edges in a bipartite graph that contains no matching with k edges and no star with l edges. (Your answer should provide a construction and prove it is best possible.)

5. Connectivity and perfect matchings.

   (a) Let G be an r-connected graph of even order having no $K_{1,r+1}$ as an induced subgraph. Prove that G has a 1-factor.

   (b) For each r, construct an r-connected graph of even order that does not contain an induced copy of $K_{1,r+3}$ and has no 1-factor.

   (Comment: this leaves unresolved whether every r-connected graph of even order without an induced copy of $K_{1,r+2}$ has a 1-factor. Note: when the number of vertices is even, the inclusion bigraph between $(r-1)$-sets and r-sets in $[2r]$ is a candidate for a sharpness example. This graph has no induced $K_{1,r+2}$ and no perfect matching. Probably it is r-connected. Can you prove it?)

6. Let v be a vertex of a 2-connected graph G. Prove that v has a neighbor u such that $G-u-v$ is connected. Find a 2-edge-connected graph G that has a vertex v such that for each neighbor u of v, the graph $G-u-v$ is disconnected.