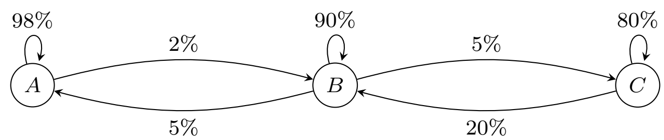
Name: Solutions

Directions: Show all work. No credit for answers without work.

1. Three cities, A, B, and C share a population. Each year, a certain percentage of people living in a city stay or move to another city according to the diagram below. For example, in a year, 90% of the population of city B stays in city B, 5% moves to city A, and 5% moves to city C.



(a) **[10 points]** Given the initial population vector $\mathbf{x_0} = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$, find a matrix A such that the populations after one year are given by $\mathbf{x_1} = A\mathbf{x_0}$.

$$a_1 = 0.98a_0 + 0.05b_0$$

$$b_1 = 0.02a_0 + 0.9b_0 + 0.2c_0$$

$$c_1 = 0.05b_0 + 0.8c_0$$

$$\text{So } A = \boxed{\begin{bmatrix} 0.98 & 0.05 & 0 \\ 0.02 & 0.9 & 0.2 \\ 0 & 0.05 & 0.8 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 98 & 5 & 0 \\ 2 & 90 & 20 \\ 0 & 5 & 80 \end{bmatrix}}$$

(b) **[15 points]** Given that 15 million people in total live in cities A, B, and C, ind equilibrium populations (if they exist).

We want $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $A\vec{x} = \vec{x}$. Note $A\vec{x} = \vec{x}$ iff $A\vec{x} - I\vec{x} = 0$ iff $(A - I)\vec{x} = \vec{0}$.

$$\text{So } A - I = A - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -2 & 5 & 0 \\ 2 & -10 & 20 \\ 0 & 5 & -20 \end{bmatrix} \sim \frac{1}{100}\begin{bmatrix} -2 & 5 & 0 \\ 0 & -5 & 20 \\ 0 & 5 & -20 \end{bmatrix}$$

$$\sim \frac{1}{100}\begin{bmatrix} -2 & 5 & 0 \\ 0 & -5 & 20 \\ 0 & 0 & 0 \end{bmatrix} \sim \frac{1}{100}\begin{bmatrix} -2 & 5 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \frac{1}{100}\begin{bmatrix} -2 & 0 & 20 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \frac{1}{100}\begin{array}{c} \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} a & b & \underset{\text{free}}{c} \end{matrix} \end{array}$$

so $x = \begin{bmatrix} 10c \\ 4c \\ c \end{bmatrix}$.

We know $10c + 4c + c = 15$ million
$15c = 15$ million
$\Rightarrow c = 1$ million.

So the equilibrium populations are $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boxed{\begin{bmatrix} 10 \text{ million} \\ 4 \text{ million} \\ 1 \text{ million} \end{bmatrix}}$

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transform that maps $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.

   (a) **[10 points]** Determine the image of $\begin{bmatrix} -1 \\ 10 \end{bmatrix}$ under T.

$$\begin{bmatrix} 1 & -3 & -1 \\ 4 & 2 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -1 \\ 0 & 14 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -1 \\ 10 \end{bmatrix}\right) = T\left(2\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) + T\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right) = 2\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 12 \\ -2 \\ 3 \end{bmatrix}}.$$

   (b) **[5 points]** How many rows and columns does the standard matrix for T have?

Standard matrix: $\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \\ \square & \square \end{bmatrix}$ So the standard matrix has $\boxed{\text{3 rows and 2 columns}}$.

It's shape is 3×2.

3. **[10 points]** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Express A as the product of elementary matrices.

Row reduce to get $E_k \cdots E_1 A = I_2$, then $A = E_1^{-1} \cdots E_k^{-1}$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 + (-3)R1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 \cdot (-\frac{1}{2})} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R1 \leftarrow R1 + (-2)R2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ $E_3^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

So $A = E_1^{-1} E_2^{-1} E_3^{-1} = \boxed{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}$

Note: other answers possible

4. **[10 points]** Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a linearly dependent set, but removing any single vector in the set gives a linearly independent set of size $p-1$. Let $c_1, \ldots, c_p$ be scalars such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. What can you say about $c_1, \ldots, c_p$? Explain.

$\boxed{\text{Either all coefficients are zero or none of them are.}}$ If $c_i = \vec{0}$, then

$$c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_p\vec{v}_p = \vec{0}$$

holds. Since $\{\vec{v}_1, \ldots, \vec{v}_p\} - \{\vec{v}_i\}$ is linearly independent, it must be that

$$c_1 = c_2 = \cdots = c_{i-1} = c_{i+1} = \cdots = c_p = 0$$

also.

5. **[15 points]** Find the inverse of the following matrix.

$$\begin{bmatrix} 1 & 2 & -1 \\ -6 & -14 & -7 \\ 2 & 5 & 4 \end{bmatrix}$$

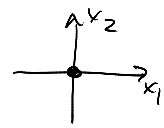
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -6 & -14 & -7 & 0 & 1 & 0 \\ 2 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3 \pm (-2)R1]{R2 \pm 6R1} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & -13 & 6 & 1 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \\ 0 & -2 & -13 & 6 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3 \pm 2R2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \\ 0 & 0 & -1 & 2 & 1 & 2 \end{array}\right] \xrightarrow{R3 \cdot (-1)} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 & -1 & -2 \end{array}\right] \xrightarrow[R2 \pm -6R3]{R1 \pm R3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & -1 & -2 \\ 0 & 1 & 0 & 10 & 6 & 13 \\ 0 & 0 & 1 & -2 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R1 \pm (-2)R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -21 & -13 & -28 \\ 0 & 1 & 0 & 10 & 6 & 13 \\ 0 & 0 & 1 & -2 & -1 & -2 \end{array}\right]$$

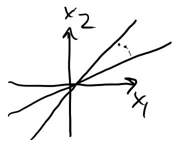
So $A^{-1} = \boxed{\begin{bmatrix} -21 & -13 & -28 \\ 10 & 6 & 13 \\ -2 & -1 & -2 \end{bmatrix}}$

6. **[5 points]** Give geometric descriptions of all subspaces of $\mathbb{R}^2$.



The zero vector forms a subspace $\{\vec{0}\}$ of dimension 0.

The 1-dimensional subspaces are all the lines in $\mathbb{R}^2$ that contain $\vec{0}$:



All of $\mathbb{R}^2$ forms a subspace of dimension 2.

7. **[2 parts, 10 points each]** Let A and B be the given matrices below. We are given that A and B are row-equivalent.

$$A = \begin{bmatrix} 16 & -3 & -47 & 60 & 10 \\ 1 & 0 & -2 & 3 & 0 \\ 5 & 4 & 10 & -1 & 3 \\ -5 & 1 & 15 & -19 & -3 \end{bmatrix} \qquad B = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 5 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \text{basic} & \text{basic} & \text{free} & \text{free} & \text{basic} \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

(a) Find a basis for Col(A).

A basis is the pivot columns of A:

$$\boxed{\left\{ \begin{bmatrix} 16 \\ 1 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 3 \\ -3 \end{bmatrix} \right\}}$$

(b) Find a basis for Nul(A).

$$\vec{x} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{bmatrix} 2x_3 - 3x_4 \\ -5x_3 + 4x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

x_3 x_4

So $\boxed{\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for Nul}(A).}$