Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 points]** Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$. Given $\mathbf{x} = \begin{bmatrix} -17 \\ 2 \\ 11 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$ if possible.

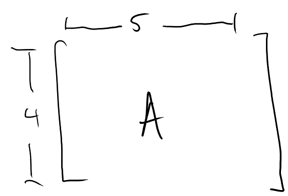
$$\left[\begin{array}{cc|c} 1 & 5 & -17 \\ -2 & -2 & 2 \\ 5 & 1 & 11 \end{array}\right] \sim \begin{bmatrix} 1 & 5 & -17 \\ 0 & 8 & -32 \\ 0 & -24 & 96 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -17 \\ 0 & 8 & -32 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -17 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

So $3\vec{b}_1 + (-4)\vec{b}_2 = \vec{x}$ and therefore

$$[\vec{x}]_{\mathcal{B}} = \boxed{\begin{bmatrix} 3 \\ -4 \end{bmatrix}}.$$

2. **[1 point]** What is the rank of a 4×5 matrix whose null space has dimension 3?



$$\text{rank}(A) + \dim(\text{nul}(A)) = \#\text{cols}$$
$$\text{rank}(A) + 3 = 5$$
$$\boxed{\text{rank}(A) = 2}$$

3. **[1 point]** Let A be an $n \times n$ matrix with two equal rows. What, if anything, can we conclude about $\det(A)$? Explain.

If A has 2 equal rows, then a row replacement operation can produce a row of all zeros. So the reduced row echelon form of A has a row of all zeros, meaning A is not row-equivalent to I_n and so A is not invertible. Since A is not invertible, we have $\boxed{\det(A) = 0.}$

4. Compute the determinant of the following matrices.

   (a) **[1 point]** $\begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$

$$2 \cdot 3 - (-1)(5) = 6 + 5 = \boxed{11}$$

(b) **[1 point]** $\begin{bmatrix} 1 & -1 & 4 \\ 1 & 3 & -2 \\ 4 & 7 & -1 \end{bmatrix}$

Soln 1 : $(1)(3)(-1) + (-1)(-2)(4) + (4)(1)(7) - (4)(3)(4) - (7)(-2)(1) - (-1)(1)(-1) = -3 + 8 + 28 - 48 + 14 = -20 + 14 - 1 + 5 = \boxed{-2}$

Soln 2 : $\begin{bmatrix} 1 & -1 & 4 \\ 0 & 4 & -6 \\ 0 & 11 & -17 \end{bmatrix} \xrightarrow{\cdot 3} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 12 & -18 \\ 0 & 11 & -17 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -1 \\ 0 & 11 & -17 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & -6 \end{bmatrix}$

$3 \det(A) = -6 \Rightarrow \det(A) = \boxed{-2}$

(c) **[2 points]** $\underbrace{\begin{bmatrix} 9 & 0 & 1 & 4 \\ 2 & -1 & 5 & 3 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 3 & -2 \end{bmatrix}}_{A}$ Cofactor expansion

$\det(A) = (-1)(+1) \det \begin{bmatrix} 9 & 1 & 4 \\ 0 & 0 & 2 \\ 1 & 3 & -2 \end{bmatrix} = -\left((2)(-1) \det \begin{pmatrix} 9 & 1 \\ 1 & 3 \end{pmatrix} \right) = 2\left(9 \cdot 3 - (1)(1)\right) = 2(26) = \boxed{52}$

(d) **[2 points]** $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 5 & 3 & 2 & 2 \\ -1 & 1 & 3 & 2 \\ -2 & 5 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -2 & -8 & -13 \\ 0 & 2 & 5 & 5 \\ 0 & 7 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -2 & -8 & -13 \\ 0 & 0 & -3 & -8 \\ 0 & 1 & -18 & -32 \end{bmatrix}$

$\xrightarrow{(-1)} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -18 & -32 \\ 0 & 0 & -3 & -8 \\ 0 & -2 & -8 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -18 & -32 \\ 0 & 0 & -3 & -8 \\ 0 & 0 & -44 & -77 \end{bmatrix} \xrightarrow{\cdot\left(-\frac{1}{11}\right)} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -18 & -32 \\ 0 & 0 & -3 & -8 \\ 0 & 0 & 4 & 7 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -18 & -32 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 4 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -18 & -32 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 11 \end{bmatrix}$

$(-1) \cdot \left(-\frac{1}{11}\right) \cdot \det(A) = 11$

$\det(A) = (11)^2 = \boxed{121}$