Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[4 parts, 1 point each]** True/False. Justify your answers.

   (a) Let A be an $(m \times n)$ matrix and let $\mathbf{x}$ be a vector. The product $A\mathbf{x}$ is defined only if $\mathbf{x}$ has size n.

True. Since A is an $(m\times n)$-matrix, A has m rows and n columns. The product $A\vec{x}$ represents a linear combination of columns of A, with weights from the components of $\vec{x}$. So the size of $\vec{x}$ must equal the number of columns of A.

   (b) If $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_p$, then so is $-\mathbf{b}$.

True. If $c_1\vec{a}_1 + \cdots + c_p\vec{a}_p = b$, then $(-c_1)\vec{a}_1 + \cdots + (-c_p)\vec{a}_p = -b$.

   (c) For all vectors $\mathbf{u}$ and $\mathbf{v}$, the set Span$\{\mathbf{u}, \mathbf{v}\}$ is bigger than the set Span$\{\mathbf{u}\}$.

FALSE. For example, if $\vec{v}$ is the zero vector, or if $\vec{v} = \vec{u}$, or if $\vec{v} = c\vec{u}$ for some scalar c, then Span$\{\vec{u}\}$ = Span$\{\vec{u}, \vec{v}\}$.

   (d) In most cases, when we choose two vectors $\mathbf{u}$ and $\mathbf{v}$ from $\mathbb{R}^3$, the sets Span$\{\mathbf{u}\}$ and Span$\{\mathbf{v}\}$ do not intersect.

False. Since the zero vector belongs to the span of every set, Span$\{\vec{u}\}$ a Span$\{\vec{v}\}$ intersect in at least the zero vector.

2. **[2 points]** For $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 8 & 2 \\ 1 & 0 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 18 \\ 24 \\ 11 \end{bmatrix}$, solve $A\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 2 & 5 & -1 & 18 \\ 3 & 8 & 2 & 24 \\ 1 & 0 & -5 & 11 \end{bmatrix} \xrightarrow{R1 \Leftrightarrow R3} \begin{bmatrix} 1 & 0 & -5 & 11 \\ 3 & 8 & 2 & 24 \\ 2 & 5 & -1 & 18 \end{bmatrix} \xrightarrow[R3 \pm (-2)R1]{R2 \pm (-3)R1} \begin{bmatrix} 1 & 0 & -5 & 11 \\ 0 & 8 & 17 & -9 \\ 0 & 5 & 9 & -4 \end{bmatrix}$$

$$\xrightarrow[\text{(avoid fractions!)}]{R2 \pm (-R3)} \begin{bmatrix} 1 & 0 & -5 & 11 \\ 0 & 3 & 8 & -5 \\ 0 & 5 & 9 & -4 \end{bmatrix} \xrightarrow{R3 \pm (-1)R2} \begin{bmatrix} 1 & 0 & -5 & 11 \\ 0 & 3 & 8 & -5 \\ 0 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{R2 \pm (-1)R3} \begin{bmatrix} 1 & 0 & -5 & 11 \\ 0 & 1 & 7 & -6 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R3 \pm (-2)R2} \begin{bmatrix} 1 & 0 & -5 & 11 \\ 0 & 1 & 7 & -6 \\ 0 & 0 & -13 & 13 \end{bmatrix} \xrightarrow{R3 \cdot (-\frac{1}{13})} \begin{bmatrix} 1 & 0 & -5 & 11 \\ 0 & 1 & 7 & -6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[R2 \pm (-7)R3]{R1 \pm 5R3} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

So the soln to $A\vec{x} = \vec{b}$ is $\vec{x} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$.

3. **[2 parts, 2 points each]** Decide whether the vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \dots \mathbf{a}_p$ given below.

(a) $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix}$

2(a)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 6 \\ 1 & -3 & -2 & 9 \end{bmatrix} \xrightarrow[R3 \pm (-1)R1]{R2 \pm (-1)R1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & 2 \\ 0 & -5 & -5 & 5 \end{bmatrix} \xrightarrow[R3 \cdot (-\frac{1}{5})]{R2 \cdot (-\frac{1}{2})} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R3 \pm (-1)R2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last column is not a pivot, the system is consistent and $\boxed{\vec{b} \text{ is a lin. comb of } \vec{a}_1, \vec{a}_2, \vec{a}_3}$.

(b) $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -8 \\ -2 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

2(b)

$$\begin{bmatrix} 1 & 7 & -8 & 3 \\ 5 & 3 & -2 & 0 \\ 2 & -2 & 3 & 1 \end{bmatrix} \xrightarrow[R3 \pm (-2)R1]{R2 \pm (-5)R1} \begin{bmatrix} 1 & 7 & -8 & 3 \\ 0 & -32 & 38 & -15 \\ 0 & -16 & 19 & -5 \end{bmatrix} \xrightarrow[R2 \cdot (-1),\ R3 \cdot (-1)]{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 7 & -8 & 3 \\ 0 & 16 & -19 & 5 \\ 0 & 32 & -38 & 15 \end{bmatrix} \xrightarrow{R3 \pm (-2)R2} \begin{bmatrix} 1 & 7 & -8 & 3 \\ 0 & 16 & -19 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Since the last column is a pivot, the system is inconsistent and so $\boxed{\vec{b} \text{ is not a lin comb. of } \vec{a}_1, \vec{a}_2, \vec{a}_3}$.