Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 parts, 2 points each]** Find a the characteristic polynomial and eigenvalues of the matrices below.

(a) $\begin{bmatrix} 12 & 10 \\ -5 & -3 \end{bmatrix}$

$$\begin{vmatrix} 12-\lambda & 10 \\ -5 & -3-\lambda \end{vmatrix} = (12-\lambda)(-3-\lambda) - (-5)(10) = \lambda^2 + 3\lambda - 12\lambda - 36 + 50$$

$$= \lambda^2 - 9\lambda + 14 = \boxed{(\lambda-7)(\lambda-2)}$$

So the eigenvalues are $\boxed{\lambda = 2 \text{ and } \lambda = 7}$.

(b) $\begin{bmatrix} 28 & 0 & 11 \\ 6 & -2 & 3 \\ -66 & 0 & -27 \end{bmatrix}$

$$\begin{vmatrix} 28-\lambda & 0 & 11 \\ 6 & -2-\lambda & 3 \\ -66 & 0 & -27-\lambda \end{vmatrix} = (28-\lambda)(-2-\lambda)(-27-\lambda) + 0 + 0 - ((-66)(-2-\lambda)(11) + 0 + 0)$$

$$= (-2-\lambda)\left[(28-\lambda)(-27-\lambda) - (-66)(11)\right]$$

$$= (\lambda+2)\left[(28-\lambda)(\lambda+27) + (-66)(11)\right]$$

$$= (\lambda+2)\left[-\lambda^2 - 27\lambda + 28\lambda + 27\cdot 28 - 6\cdot 11\cdot 11\right]$$

$$= (\lambda+2)\left[-\lambda^2 + \lambda + (3\cdot 9)\cdot(2\cdot 14) - 6\cdot 11^2\right]$$

$$= (\lambda+2)\left[-\lambda^2 + \lambda + 6\cdot(9\cdot 14 - 11^2)\right]$$

$$= (\lambda+2)\left[-\lambda^2 + \lambda + 6\cdot(126 - 121)\right]$$

$$= (\lambda+2)\left[-\lambda^2 + \lambda + 6\cdot 5\right] = -(\lambda+2)(\lambda^2 - \lambda - 30)$$

$$= \boxed{-(\lambda+2)(\lambda-6)(\lambda+5)}$$

So the eigenvalues are $\boxed{\lambda = 6,\ \lambda = -2, \text{ and } \lambda = -5.}$

2. **[2 points]** Find a basis for the eignespace associated with eigenvalue $\lambda = 2$ for the matrix given below.

$$\begin{bmatrix} -1 & -1 & 1 & -2 \\ 8 & 5 & -2 & 5 \\ -2 & -1 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -3 & -1 & 1 & -2 \\ 8 & 3 & -2 & 5 \\ -2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & -1 & 2 \\ 8 & 3 & -2 & 5 \\ -2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 8 & 3 & -2 & 5 \\ -2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & 6 & -3 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & \underbrace{x_3 \quad x_4}_{\text{free}} \end{matrix} \end{array}$$

$$\vec{x} = \begin{bmatrix} 1x_3 - 1x_4 \\ -2x_3 + 1x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

So a basis is $\boxed{\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$.

3. **[2 points]** Let $f(\lambda)$ be the characteristic polynomial of the $n \times n$ matrix A, and let h be a scalar. Find the characteristic polynomial of the matrix $A + hI$ in terms of f.

Char. poly of $A + hI$: $\det((A + hI) - \lambda I) = \det(A + (h - \lambda)I)$

$= \det(A - (\lambda - h)I) = \det(A - tI) = f(t)$, where $t = \lambda - h$.

So the char. polynomial of $A + hI$ is $f(t)$ or $\boxed{f(\lambda - h)}$.

4. **[2 points]** Is there an $n \times n$ matrix A such that the eigenspace associated with eigenvalue $\lambda = 3$ is all of $\mathbb{R}^n$? Either give an example or explain why not.

Yes: let $\boxed{A = 3I.}$ Then $\text{Nul}(A - 3I) = \text{Nul}(3I - 3I) = \text{Nul}(0) = \mathbb{R}^n$.

$(n \times n)$ all zeros matrix