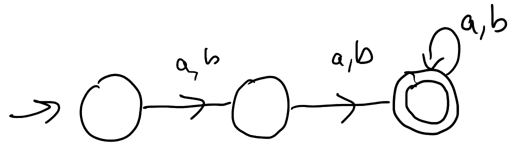
Name: Solutions

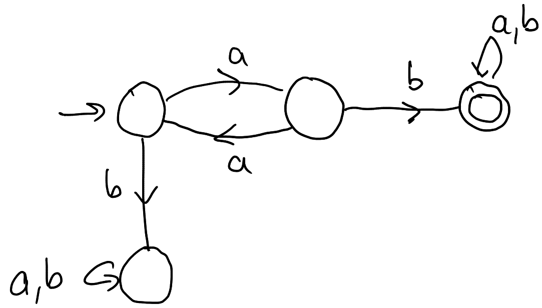
**Directions:** Show all work. Answers without work generally do not earn points. Unless stated otherwise, answers may be left in terms of factorials and binomial coefficients. Your answers should not involve sums with more than a few terms.

1. [**3 parts, 6 points each**] Let $\Sigma = \{a, b\}$. Construct DFAs for the following languages.

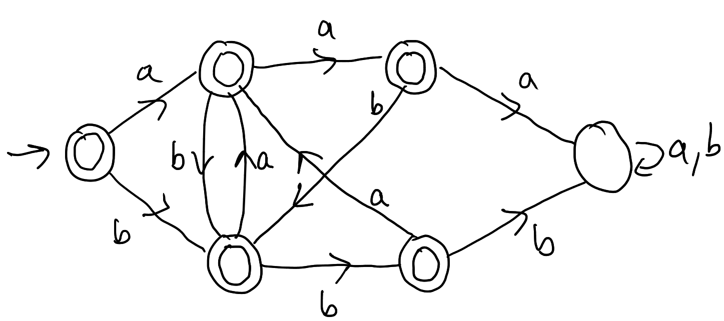
   (a) $\{w \in \Sigma^* \colon\ w \text{ has length at least } 2\}$




   (b) $\{w \in \Sigma^* \colon\ w = a^n bx \text{ for some odd integer } n \text{ and some string } x\}$



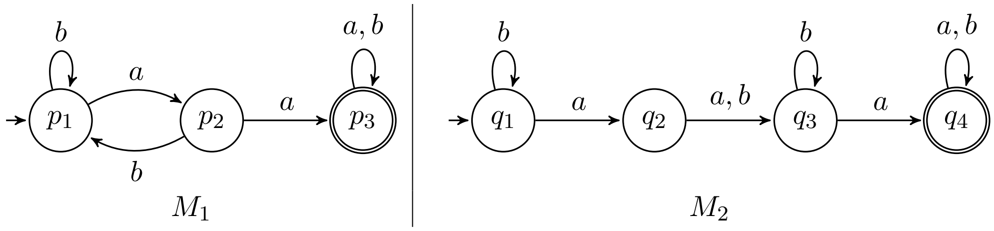

   (c) $\{w \in \Sigma^* \colon\ w \text{ does not contain the same symbol three times consecutively}\}$




2. [**1 point**] Fill in the blanks: In the formal definition of a DFA, the transition function $\delta$ has type

$$\delta\colon \underline{\quad Q \times \Sigma \quad} \to \underline{\quad Q \quad}.$$

3. Let $\Sigma = \{a, b\}$. The DFAs $M_1$ and $M_2$ are pictured below.

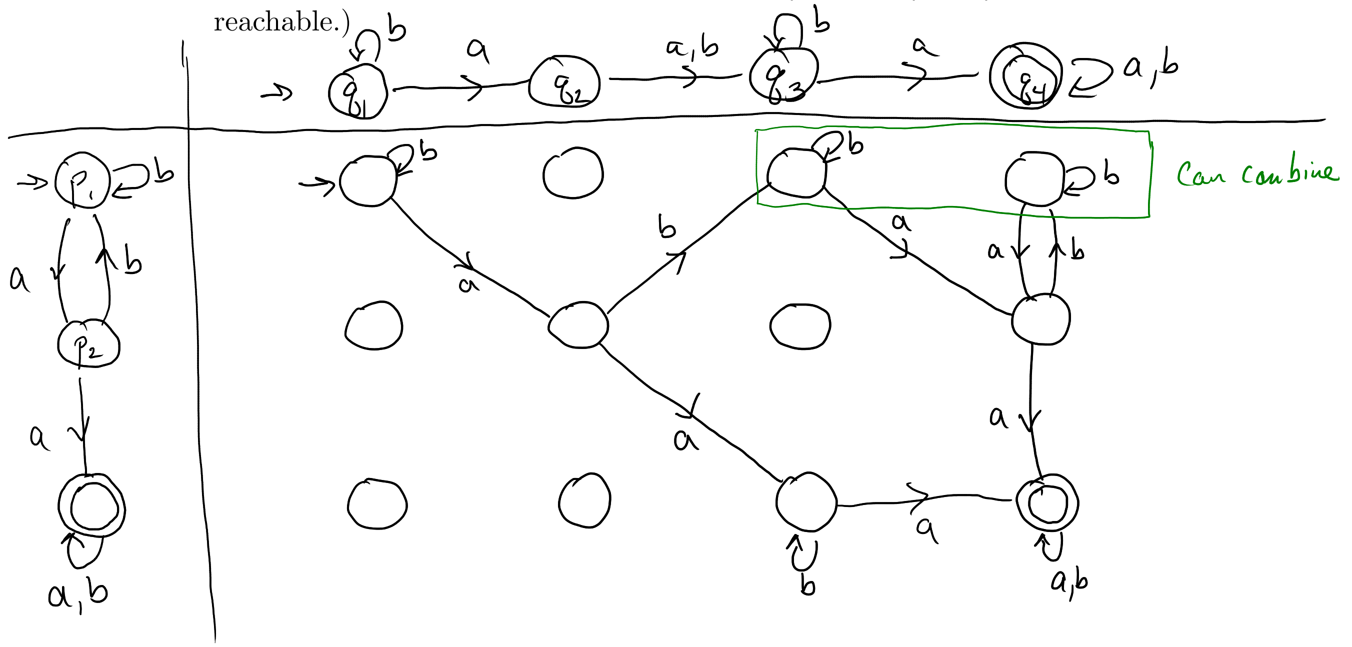



(a) [**6 points**] Give simple, English descriptions of the languages computed by $M_1$ and $M_2$.

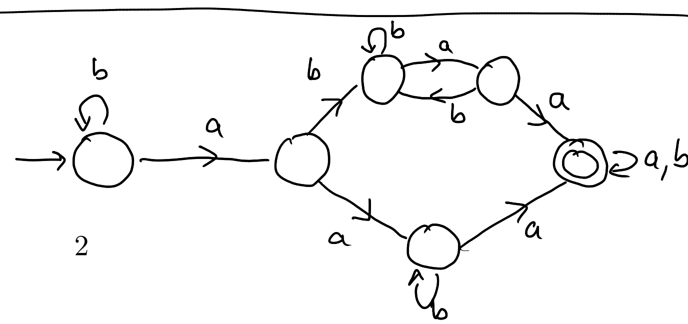
$L(M_1)$ is the set of strings with two consecutive a's

$L(M_2)$ is the set of strings with a pair of non-consecutive a's.

(b) [**10 points**] Construct a DFA that accepts a string $w$ if and only if both $M_1$ and $M_2$ accept $w$. (Hint: since many states would be unreachable in the product construction, it is more efficient to find successor states dynamically, only for those states that are reachable.)




Redraw and simplify (optional):



4. **[6 parts, 3 points each]** True/False. For each statement below, decide if the statement is true or false, and justify your answer.

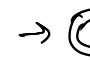
   (a) The language $A$ is regular if and only if $A$ is computed by some DFA.

   $\boxed{\text{True}}$: this is our definition of a regular language

   (b) The language $A$ is regular if and only if $A$ is computed by some NFA.

   $\boxed{\text{True}}$: If $A$ is regular, then it is computed by a DFA, which is an NFA. Conversely, if $A$ is computed by an NFA $N$, then some DFA $M$ is equivalent to $N$ and also computes $A$.

   (c) If $A$ is the language computed by an NFA $N$, then we obtain an NFA for the complement language $\overline{A}$ by inverting the accepting and rejecting states in $N$.

   $\boxed{\text{FALSE}}$ Let $\Sigma = \{a, b\}$, and define $N_1$:  || $N_2$ 

   We have $L(N_1) = \{\lambda\}$ and $L(N_2) = \emptyset$ which are not complements of one another.

   (d) Let $\Sigma = \{a, b\}$. There is a DFA that computes the language $\{a^n b^n : \ n \geq 0\}$.

   $\boxed{\text{FALSE.}}$ We have seen in class that this language is not regular.

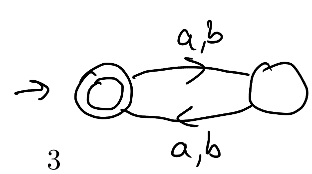
   (e) If $A$ and $B$ are regular languages, then $A - B$ is also a regular language.

   $\boxed{\text{True}}$. Given DFAs $M_1$ and $M_2$ for $A$ and $B$, the product construction gives a way to construct a DFA for $A - B$.

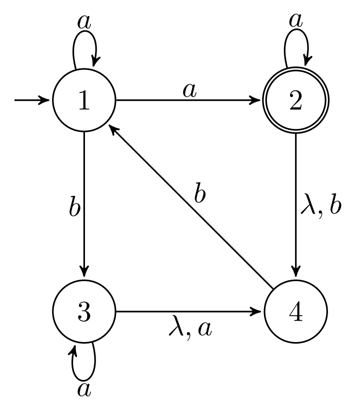
   (f) Let $\Sigma = \{a, b\}$. There is a DFA that computes the language

   $$\{w \in \Sigma^* : \ w = xy \text{ for some strings } x \text{ and } y \text{ with } |x| = |y|\}.$$

   (Recall that if $x$ is a string in $\Sigma^*$, then $|x|$ denotes the length of $x$.)

   $\boxed{\text{True}}$. This language is also just $\{w \in \Sigma^* : |w| \text{ is even}\}$, and is computed by .

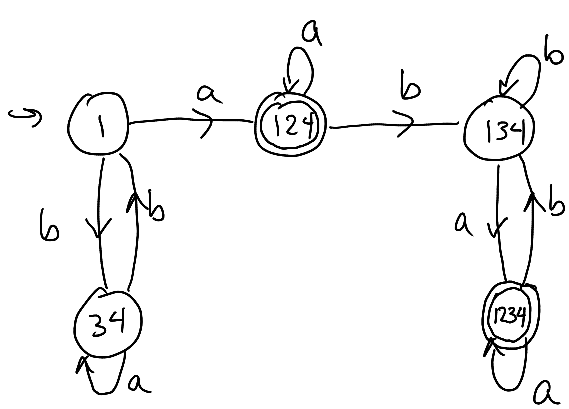
5. Let $N$ be the NFA pictured below.



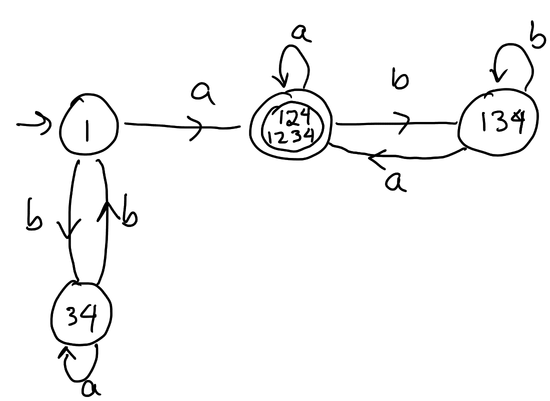
(a) [**6 points**] Complete the transition table below.

| | $\lambda^*$ | $\lambda^* a$ | $\lambda^* a \lambda^*$ | $\lambda^* b$ | $\lambda^* b \lambda^*$ |
|---|---|---|---|---|---|
| 1 | 1 | 12 | 124 | 3 | 34 |
| 2 | 24 | 2 | 24 | 14 | 14 |
| 3 | 34 | 34 | 34 | 1 | 1 |
| 4 | 4 | $\emptyset$ | $\emptyset$ | 1 | 1 |

(b) [**10 points**] Convert $N$ to a DFA.

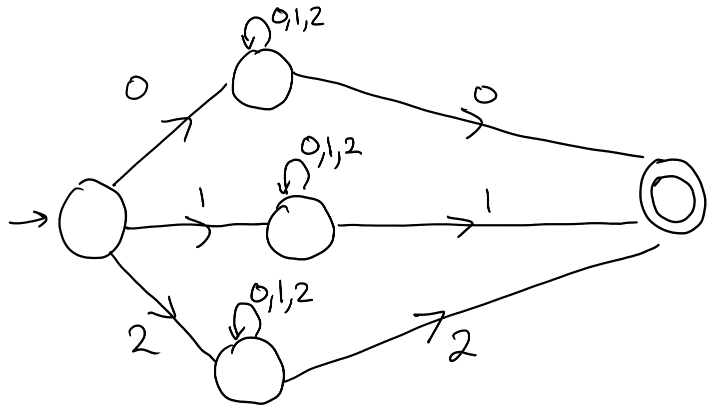


Alt answer, simplified (124, 1234 combined)

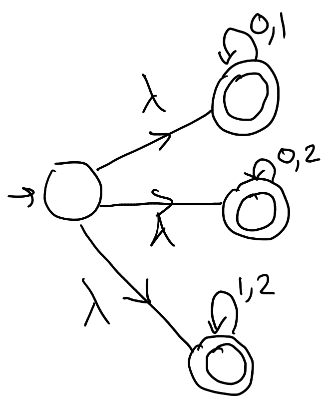


6. **[2 parts, 5 points each]** Let $\Sigma = \{0, 1, 2\}$. Construct NFAs for the following languages with at most the prescribed number of states.

   (a) A 5-state NFA for $\{w \in \Sigma^* : |w| \geq 2 \text{ and starts and ends with the same symbol}\}$



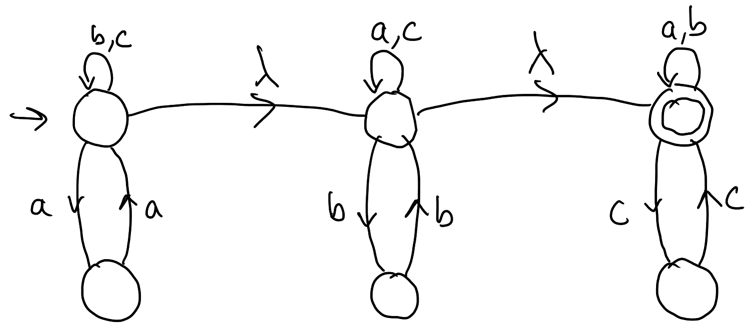

   (b) A 4-state NFA for $\{w \in \Sigma^* : w \text{ does not contain all 3 input symbols}\}$




7. **[5 points]** Let $\Sigma = \{a, b, c\}$ and let

$$A = \{w \in \Sigma^* : \#a(w) \text{ is even}\}$$
$$B = \{w \in \Sigma^* : \#b(w) \text{ is even}\}$$
$$C = \{w \in \Sigma^* : \#c(w) \text{ is even}\}.$$

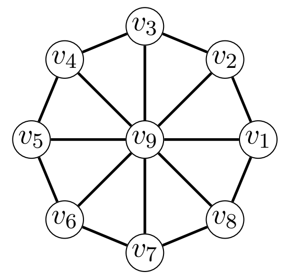
Construct an NFA for the language $ABC$, where $ABC = \{xyz \in \Sigma^* : x \in A,\ y \in B, \text{ and } z \in C\}$. (Do not convert to a DFA.)



8. [**4 points**] How many edges are there in $K_{50}$, the complete graph on 50 vertices?

$$\boxed{\binom{50}{2}} \quad \text{or} \quad \frac{50 \cdot 49}{2} = 25 \cdot 49 = \boxed{1225}$$

9. [**3 parts, 4 points each**] Let $G$ be the following graph with 9 vertices.




(a) What are the degrees of the vertices in $G$?

Vertices in $\{v_1, \ldots, v_8\}$ have degree 3.

The vertex $v_9$ has degree 8.

(b) How many times does $C_3$ appear as a subgraph of $G$?

There are $\boxed{8}$ copies of $C_3$: each edge in the outer 8-cycle completes one copy of $C_3$ with $v_9$.

(c) In total, how many cycles does $G$ contain?

$$\boxed{1 + 2\binom{8}{2}}$$

outer 8-cycle

cycles containing $v_9$:
1. Select two vertices adjacent to $v_9$: $\binom{8}{2}$
2. Choose one of two arcs along outer cycle: 2

In total we get

$$1 + 2 \cdot \frac{8 \cdot 7}{2} =$$

$$1 + 8 \cdot 7 = \boxed{57}$$

cycles.

Scratch Paper