Name: Solutions

Directions: Show all work. Answers without work generally do not earn points.

1. **[6 parts, 2 points each]** Let $A = \{2, 3, \{2, 3\}\}$, $B = \{\varnothing, 2, \{3\}\}$, and $C = \{(3, 2), (2, 3), \{2, 3\}, \{3, 2\}\}$.

   (a) Determine the sizes of sets A, B, and C.

   $|A| = 3$, $|B| = 3$, $|C| = 3$ (For C, note that $\{2,3\} = \{3,2\}$.)

   (b) True or False (write the whole word): $\{3, 2\} \in C - A$

   $\boxed{\text{FALSE}}$ $\{3,2\} = \{2,3\} \in A$

   (c) True or False (write the whole word): $(3, 2) \in C - A$

   $\boxed{\text{True}}$ $(3,2) \neq \{2,3\}$

   (d) Determine $A \cap B$.

   $A \cap B = \boxed{\{2\}}$

   (e) Determine $A \cup B$.

   Colums

   $A \cup B = \boxed{\{2, 3, \{2,3\}, \varnothing, \{3\}\}}$

   (f) Determine $\mathcal{P}(B - A)$.

   $B - A = \{\underbrace{\varnothing}_{x}, \underbrace{\{3\}}_{y}\}$, $\mathcal{P}(B-A) = \boxed{\{\varnothing, \{\underbrace{\varnothing}_{x}\}, \{\underbrace{\{3\}}_{y}\}, \{\underbrace{\varnothing}_{x}, \underbrace{\{3\}}_{y}\}\}}$

2. **[2 parts, 2 points each]** A set A of integers is *closed under addition* if $x + y \in A$ whenever $x \in A$ and $y \in A$.

   (a) Give two examples of an infinite set of integers that is closed under addition.

   $A = \{2k : k \in \mathbb{Z}\}$, the set of all even integers

   $A = \{k : k \in \mathbb{Z}, k \geq 0\}$, the set of all ~~positive~~ non-negative integers

   Many other answers possible

   (b) Which finite sets of integers, if any, are closed under addition? ~~Explain.~~ Explain

   $\boxed{\text{Only } A = \{0\}}$ $\boxed{\text{and } A = \varnothing.}$ If A contains an non-zero integer x, then $\{x, 2x, 3x, 4x, \ldots\}$ is an infinite subset of A.

Note: Since the condition allows $x = y$ we have $x + x \in A$ for all $x \in A$. Also, when $x \in A$, we have $2x \in A$ and so $x + 2x \in A$. And so on.

6

3. **[4 parts, 3 points each]** Let A, B, and C be sets. Express the following sets as concisely as possible using standard set operations. For example, "The set of all elements that are in A or B" is $A \cup B$.

   (a) The set of all elements that are in B but not A.

   $$\boxed{B - A}$$

   (b) The set of all ordered pairs of elements in C.

   $$\boxed{C^2} \quad \text{or} \quad \boxed{C \times C}$$

   (c) The set of all elements that belong to exactly one of A and C.

   $$\boxed{A \triangle C}$$

   (d) The set of all sets X such that $X \subseteq A \cup B$, $X \cap A \neq \varnothing$, and $X \cap B \neq \varnothing$.

   $$\boxed{\mathcal{P}(A \cup B) - \mathcal{P}(A - B) - \mathcal{P}(B - A)} \quad \text{or} \quad \boxed{\mathcal{P}(A \cup B) - (\mathcal{P}(A - B) \cup \mathcal{P}(B - A))}$$

4. **[2 parts, 3 points each]** Let A be a set of size n.

   (a) Determine the size of $\mathcal{P}(A \times A)$.

   $$|\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^{n \cdot n} = \boxed{2^{(n^2)}}$$

   (b) Determine the size of $\mathcal{P}(A) \times \mathcal{P}(A)$.

   $$|\mathcal{P}(A) \times \mathcal{P}(A)| = |\mathcal{P}(A)| \cdot |\mathcal{P}(A)| = 2^n \cdot 2^n = 2^{2n} = \boxed{4^n}$$

5. **[5 points]** Are there any sets whose size is larger than the set of real numbers $\mathbb{R}$? If yes, then give an example. If no, then explain why not.

Yes: $\boxed{P(R)}$ is a larger set.

6. **[5 points]** Identify a significant consequence of Cantor's argument that the real numbers are uncountable in the field of computer science.

Since the set of programs is countable but the set of all possible computational problems has the same size as R, there are computational problems that cannot be solved algorithmically

7. **[6 points]** Let A and B be countable sets. Argue that $A \times B$ is countable.

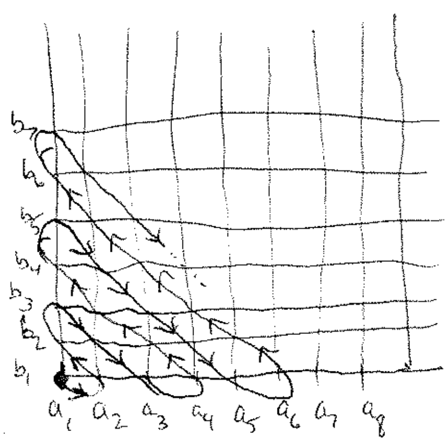
Since A is countable, we can list the elements of A:

$a_1, a_2, a_3, a_4, \ldots$

Since B is countable, we can list the elements of B:

$b_1, b_2, b_3, b_4, \ldots$

To show that $A \times B$ is countable, we list the elements of $A \times B$ using the same "zig-zag" strategy that we used for $\mathbb{N} \times \mathbb{N}$:



$A \times B$: (a_1, b_1),
(a_2, b_1), (a_1, b_2),
(a_1, b_3), (a_2, b_2), (a_3, b_1),
(a_4, b_1), (a_3, b_2), (a_2, b_3), (a_1, b_4),
$\vdots$

16

8. **[3 parts, 4 points each]** A class contains 8 men and 10 women. A group of 6 students is chosen at random.

(a) What is the probability that all students in the chosen group are men?

$$\Pr(\text{all men}) = \boxed{\frac{\binom{8}{6}}{\binom{18}{6}}} = \frac{28}{18564} = \boxed{\frac{1}{663}} \approx 0.00151$$

(b) What is the probability that all students in the chosen group are women?

$$\Pr(\text{all women}) = \boxed{\frac{\binom{10}{6}}{\binom{18}{6}}} = \frac{210}{18564} = \boxed{\frac{5}{442}} \approx 0.01131$$

(c) What is the probability that the chosen group has at least one man and at least one woman?

$$\Pr(\geq 1 \text{ man and } \geq 1 \text{ woman}) = 1 - \Pr(\text{all men}) - \Pr(\text{all women})$$

$$= \boxed{1 - \frac{\binom{8}{6}}{\binom{18}{6}} - \frac{\binom{10}{6}}{\binom{18}{6}}} = \boxed{\frac{77}{78}} \approx 0.98718$$

9. **[4 points]** Two different cards are drawn at random from a deck of 12 cards, each labeled with an integer in $\{1, \ldots, 12\}$. What is the probability that the difference between the values on the chosen cards is at most 3?

$$\#\text{pairs with distance } 1 = |\{(1,2), (2,3), \ldots, (11,12)\}| = 11$$

$$\#\text{pairs with distance } 2 = |\{(1,3), (2,4), \ldots, (10,12)\}| = 10$$

$$\#\text{pairs with distance } 3 = |\{(1,4), (2,5), \ldots, (9,12)\}| = 9$$

$$\Pr(\text{Distance} \leq 3) = \frac{\#\text{pairs dist} \leq 3}{\text{total } \#\text{ pairs}} = \frac{11+10+9}{\binom{12}{2}} = \frac{30}{\frac{12 \cdot 11}{2}} = \frac{30}{66} = \boxed{\frac{15}{33}} = \boxed{\frac{5}{11}}$$

10

10. Suppose that a pair of dice are rolled. Let A be the event that the two rolled values are the same, let B be the event that the sum is in $\{6, 7, 8\}$, and let C be the event that both values rolled are at least 4.

(a) **[3 points]** Give the sample space Ω.

$$\Omega = \{1,2,3,4,5,6\}^2$$

(b) **[6 points]** Find $\Pr(A)$, $\Pr(B)$, and $\Pr(C)$.

$$\Pr(A) = \frac{|A|}{|\Omega|} = \frac{6}{36} = \boxed{\frac{1}{6}}$$

$$\Pr(B) = \frac{|B|}{|\Omega|} = \frac{5+6+5}{36} = \frac{16}{36} = \boxed{\frac{4}{9}}$$

$$\Pr(C) = \frac{|C|}{|\Omega|} = \frac{3\cdot 3}{36} = \frac{9}{36} = \boxed{\frac{1}{4}}$$

(c) **[6 points]** Find $\Pr(A \cap B)$, $\Pr(B \cap C)$, and $\Pr(C \cap A)$.

$$\Pr(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{|\{(3,3),(4,4)\}|}{36} = \frac{2}{36} = \boxed{\frac{1}{18}}$$

$$\Pr(B \cap C) = \frac{|B \cap C|}{|\Omega|} = \frac{|\{(4,4)\}|}{36} = \boxed{\frac{1}{36}}$$

$$\Pr(C \cap A) = \frac{|\{(4,4),(5,5),(6,6)\}|}{36} = \frac{3}{36} = \boxed{\frac{1}{12}}$$

(d) **[3 points]** For each of the pairs of events $\{A, B\}$, $\{B, C\}$, and $\{C, A\}$, determine whether the pair is independent, positively correlated, or negatively correlated.

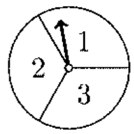
$\Pr(A \cap B) = \frac{1}{18} < \frac{1}{6}\cdot\frac{4}{9} = \Pr(A)\cdot\Pr(B)$, so A and B are $\boxed{\text{negatively correlated}}$

$\Pr(B \cap C) = \frac{1}{36} < \frac{4}{9}\cdot\frac{1}{4} = \Pr(B)\cdot\Pr(C)$, so B and C are $\boxed{\text{negatively correlated}}$

$\Pr(C \cap A) = \frac{1}{12} > \frac{1}{6}\cdot\frac{1}{4} = \Pr(C)\cdot\Pr(A)$, so A and C are $\boxed{\text{positively correlated}}$

18

11.



A spinner has three equal regions, labeled 1, 2, and 3. Each spin is equally likely to stop in each of the three regions.

(a) **[6 points]** A contestant executes 3 spins. What is the probability that each region is the result of one spin?

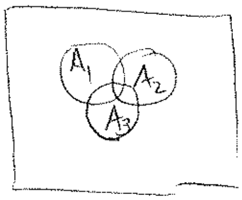
A

$\Omega = \{1,2,3\}^3$, $|\Omega| = 3^3 = 27$

$|A| = 3 \cdot 2 \cdot 1 = 6$

$Pr(A) = \frac{|A|}{|\Omega|} = \frac{6}{27} = \boxed{\frac{2}{9}}$

(b) **[6 points]** A contestant executes n spins. what is the probability that each region is the result of at least one spin? (Note: your answer should be a formula involving n which agrees with part (a) when $n = 3$.)



$\Omega = \{1,2,3\}^n$. Let A_i be the event that region i is never the result.

$$\begin{aligned} Pr(A_1 \cup A_2 \cup A_3) &= Pr(A_1) + Pr(A_2) + Pr(A_3) - Pr(A_1 \cap A_2) - Pr(A_2 \cap A_3) - Pr(A_3 \cap A_1) + Pr(A_1 \cap A_2 \cap A_3) \\ &= 3\left(\frac{2^n}{3^n}\right) - 3\left(\frac{1^n}{3^n}\right) + 0 \\ &= 3\left[\frac{2^n - 1}{3^n}\right] = \frac{2^n - 1}{3^{n-1}} \end{aligned}$$

Now $Pr(\text{all 3 regions appear}) = 1 - Pr(A_1 \cup A_2 \cup A_3) = \boxed{1 - \frac{2^n - 1}{3^{n-1}}}$.

(c) **[4 points]** A contestant executes n spins. What is the probability that the spins appear in order, with all region 1 spins happening before all region 2 spins, and all region 2 spins happening before all region 3 spins?

Let A be the event that the regions appear in order.

There is a 1 to 1 correspondence between A and integral solns to

$x_1 + x_2 + x_3 = n$.

The solution describes the number of times each region occurs.

So $|A| = \binom{n+2}{2}$.

$Pr(A) = \frac{|A|}{|\Omega|} = \boxed{\frac{\binom{n+2}{2}}{3^n}}$.

16