Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[4 parts, 1 point each]** Let $A = \{1, 2, 3\}$, let $B = \{3, 4\}$, and let $C = \varnothing$.

   (a) Determine the sets $A \times B$ and $A \times C$.

   $A \times B = \{(1,3), (2,3), (3,3), (1,4), (2,4), (3,4)\}$

   $A \times C = \varnothing$

   (b) True or False (write the whole word): $(A - B) \subseteq (A - B)^2$.

   $\boxed{\text{False}}$. The integer 1 is a member of $A-B$ but every element in $(A-B)^2$ (or $(A-B)\times(A-B)$) is an ordered pair

   (c) Give two examples of elements in $\mathcal{P}(A) \times \mathcal{P}(B)$.: Pairs of the form (subset of A, subset of B).

   Many examples: $(\varnothing, \varnothing)$, (A, B), $(\{1,3\}, \{3\})$, $(\{1\}, \{3,4\})$, ...

   (d) Give two examples of elements in $\mathcal{P}(A \times B)$.: Subsets of $A \times B$.

   For instance: $\varnothing$, $\{(1,3), (3,3)\}$, $\{(1,4), (2,4), (3,4)\}$, $A \times B$, ...

2. **[2 parts, 1 point each]** Express the following statements using concise mathematical notation. For example, the statement "The set A is a member of the set B" may be expressed as "$A \in B$".

   (a) "Every element in A is also an element in B."

   $A \subseteq B$

   (b) "Every subset of B and every subset of C is a member of the set A."

   $\mathcal{P}(B) \cup \mathcal{P}(C) \subseteq A$

3. **[2 points]** An infinite bitstring is *periodic* if it consists of repeated copies of a finite bitstring. For example, $0000\cdots$ consists of repeated copies of 0, and $011011011\cdots$ consists of repeated copies of 011. Let A be the set of periodic infinite bitstrings. Is A countable? Justify your answer.

$\boxed{\text{Yes, A is countable}}$. Begin with an enumeration of all finite, non-empty bitstrings: 0, 1, 00, 01, 10, 11, 000, 001, 010, ... (We can enumerate finite bitstrings by their lengths, as in HW5.)

~~Then~~

Next, replace each finite bitstring x with the periodic bitstring $xxxx\cdots$.

So, 0, 1, 00, 01, 10, 11, ... becomes 000000..., 11111..., ~~000000...~~, 010101..., 101010..., ~~11111...~~, ...

Finally, cross out repeated bitstrings

4. **[2 points]** A sequence $n_1, n_2, n_3, \ldots$ of positive integers is *increasing* if $n_1 < n_2 < \cdots$. Let B be the set of increasing sequences of positive integers. Is B countable? Justify your answer.

$\boxed{\text{No. B is uncountable}}$ We modify Cantors Diagonalization Argument.

Suppose for a contradiction that ~~such~~ a list $l_1, l_2, l_3, \ldots$ of all such sequences exists:

l_1: $a_{1,1} < a_{1,2} < a_{1,3} < \cdots$ (1, 2, 3)

l_2: $a_{2,1} < a_{2,2} < a_{2,3} < \cdots$ (1, 5, 6)

l_3: $a_{3,1} < a_{3,2} < a_{3,3} < \cdots$ (2, 3, 4)

l: $2 < 6 < 7 < \cdots$

Make a new list l by ~~choosing~~ setting, for each i, ~~an~~ the i^{th} entry to be an integer larger than earlier entries in l but still different from the i^{th} entry in l_i. Since l is an increasing list of positive integers and l does not appear in $l_1, l_2, \ldots$, we have a contradiction.