Solutions

Math261 **Worksheet 3** **September 9, 2014**

1. Apply the existence and uniqueness theorems to the following. What can you conclude without solving the differential equation?

   (a) $y' = t \ln y$ with $y(1) = 1$

   Non-linear. $f = t \ln y$, $\frac{\partial f}{\partial y} = \frac{t}{y}$

   Both f and $\frac{\partial f}{\partial y}$ are continuous when $y > 0$, so the solution $\boxed{\text{exists and is unique}}$.

   (b) $y' = (\ln t)y$ with $y(1) = 1$

   Linear. $y' - (\ln t)y = 0$ with $p(t) = -\ln(t)$ and $g(t) = 0$. Note: $p(t)$ is continuous for $t > 0$ and $g(t)$ is continuous everywhere. Therefore the solution $\boxed{\text{exists and is unique on } (0, \infty)}$.

   (c) $(t-6)y' + y = \sqrt{t+1}$ with $y(0) = 0$

   Linear. $y' + \frac{1}{t-6} y = \frac{\sqrt{t+1}}{t-6}$ with $p(t) = \frac{1}{t-6}$, $g(t) = \frac{\sqrt{t+1}}{t-6}$.

   - $p(t)$ is continuous everywhere except $t = 6$.
   - $g(t)$ is continuous on $t \geq -1$ except at $t = 6$.

   Therefore the $\boxed{\text{solution exists and is unique on } (-1, 6)}$.

   (d) $y' = |y|$ with $y(0) = 0$.

   Non-linear. $f = |y|$ and $\frac{\partial f}{\partial y} = \begin{cases} -1 & y < 0 \\ 1 & y > 0 \end{cases}$

   - $f(t, y)$ is continuous everywhere
   - $\frac{\partial f}{\partial y}$ is continuous everywhere except $y = 0$.

   Since the initial condition is $y(0) = 0$, and only f is continuous here, the $\boxed{\text{solution exists but uniqueness is not guaranteed}}$

2. Solve $y' + ty = ty^4$.

using the Bernoulli eqn subst. with $v = y^{1-4} = y^{-3}$.

$$y = v^{-\frac{1}{3}}, \quad \frac{dy}{dt} = -\frac{1}{3} v^{-\frac{4}{3}} \frac{dv}{dt}$$

$$-3v^{4/3}\left[-\frac{1}{3} v^{-4/3}\frac{dv}{dt} + tv^{-\frac{1}{3}} = tv^{-\frac{4}{3}}\right]$$

$$\frac{dv}{dt} - 3tv = -3t$$

$$\mu = e^{\int -3t\,dt} = e^{-\frac{3}{2}t^2}$$

$$e^{-3/2 t^2}\frac{dv}{dt} - 3te^{-\frac{3}{2}t^2} v = -3te^{-\frac{3}{2}t^2}$$

$$\frac{d}{dt}\left[e^{-\frac{3}{2}t^2} v\right] = -3te^{-\frac{3}{2}t^2}$$

$$e^{-\frac{3}{2}t^2} v = \int -3te^{-\frac{3}{2}t^2}$$

$$e^{-\frac{3}{2}t^2} v = e^{-\frac{3}{2}t^2} + C$$

$$v = 1 + Ce^{\frac{3}{2}t^2}$$

$$\frac{1}{y^3} = 1 + Ce^{\frac{3}{2}t^2}$$

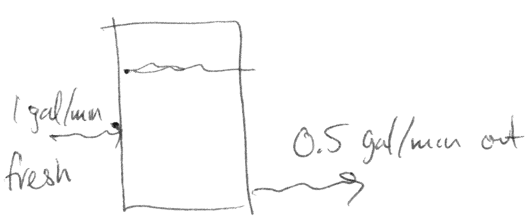
$$\boxed{y = \left[\frac{1}{1+Ce^{\frac{3}{2}t^2}}\right]^{1/3}}$$

3. A large-capacity fish tank initially contains 100 gal of water and 20 lbs of salt, but the salt concentration is too high. Fresh water is pumped in at a rate of 1 gal/min and the tank is drained at a rate of 0.5 gal/min. Assume the salt is well-mixed in the tank.

(a) Write an equation for $V(t)$, where V is the volume of water (in gal) in the tank at time t (in minutes).

$$V(t) = 100 + \frac{1}{2}t$$

(b) Write a differential equation for $Q(t)$, where Q is the quantity of salt in the tank (in lbs) at time t (in minutes).



$$\frac{dQ}{dt} = 0 - \frac{Q}{V}\cdot\frac{1}{2}$$

no salt in

$$\boxed{\frac{dQ}{dt} = -\frac{Q}{100+\frac{1}{2}t}\cdot\frac{1}{2} = \frac{-Q}{200+t}}$$

(c) Solve for $Q(t)$.

$$\int \frac{1}{Q}\,\frac{dQ}{\cancel{dt}} = -\int \frac{1}{200+t}\,dt$$

$$\ln|Q| = -\ln|200+t| + C$$

$$Q = \cancel{\pm} e^{\ln\left|\frac{1}{200+t}\right|} \cdot e^{C}$$

$$Q = C \cdot \frac{1}{200+t}$$

Impose $Q(0) = 20$:

$$20 = \frac{C}{200+0}; \quad C = 4000$$

$$\boxed{Q = \frac{4000}{200+t}}$$

(d) How long will it take for the tank to reach a concentration of 0.1 lbs of salt per gallon?

Set $\frac{Q}{V} = 0.1$ and solve for t.

$$\frac{\frac{4000}{200+t}}{100+\frac{1}{2}t} = 0.1$$

$$\frac{4 \cdot 10^3}{(200+t)(100+\frac{1}{2}t)} = \frac{1}{10}$$

$$4 \cdot 10^4 = (200+t)(100+\tfrac{1}{2}t)$$

$$4 \cdot 10^4 = 2 \cdot 10^4 + 200t + \tfrac{1}{2}t^2$$

$$t^2 + 400t + 4 \cdot 10^4 = 8 \cdot 10^4$$

$$t^2 + 400t - 4 \cdot 10^4 = 0$$

$$t = \frac{-400 \pm \sqrt{(4 \cdot 10^2)^2 + 4 \cdot 4 \cdot 10^4}}{2}$$

$$t = -200 \pm \sqrt{\tfrac{1}{4}(4^2 \cdot 10^4 + 4 \cdot 4 \cdot 10^4)}$$

$$= -200 \pm \sqrt{8 \cdot 10^4}$$

$$= -200 \pm \sqrt{2}\,(200) \approx \boxed{82.84 \text{ min}}$$

Take soln with $t \geq 0$: $\cancel{t = (\sqrt{6} - 2) \cdot 100 = \boxed{44.95 \text{ min}}}$

$t = 200(\sqrt{2} - 1) = 82.84$